

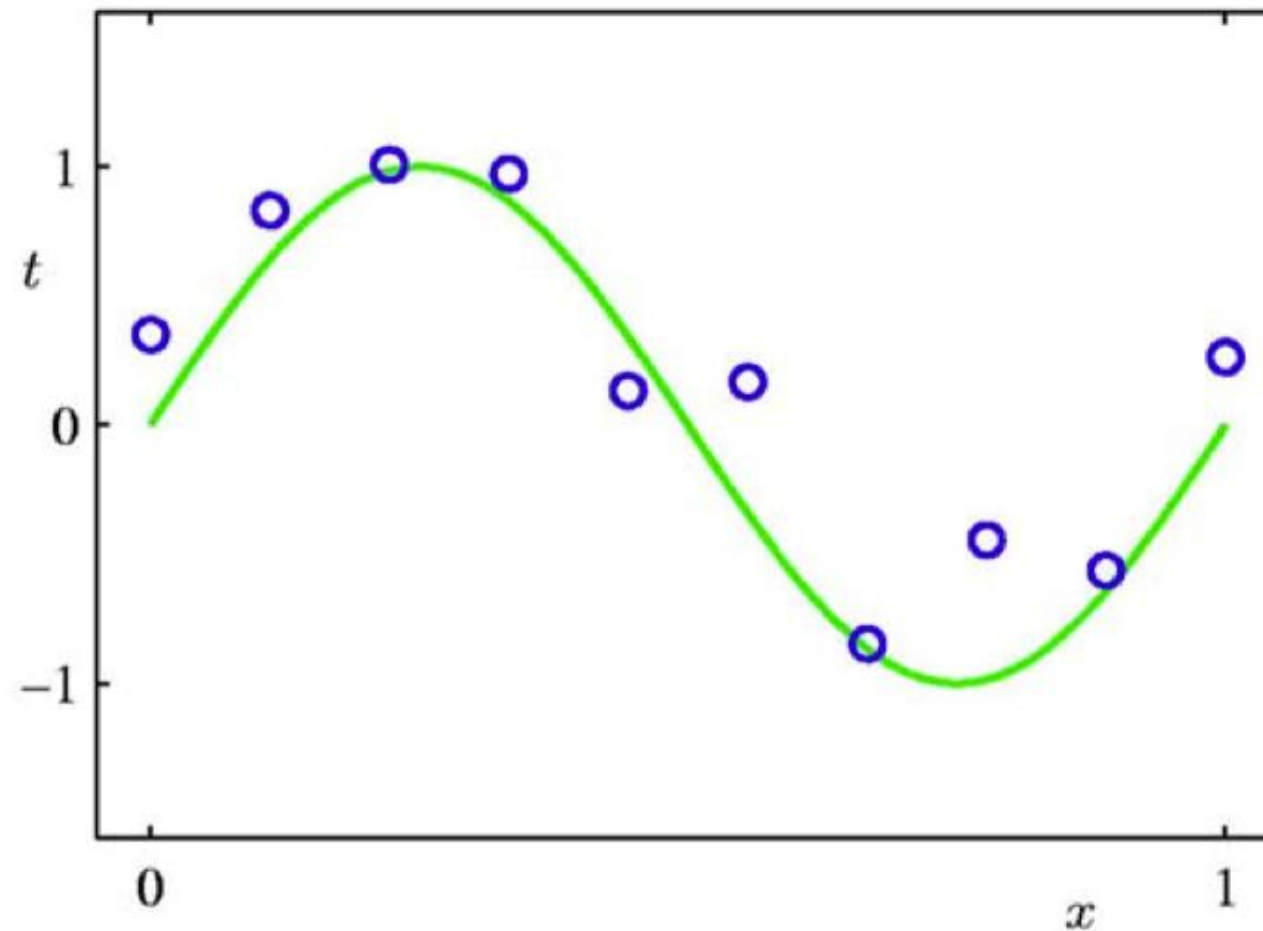
Regularized Linear Regression

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Outline

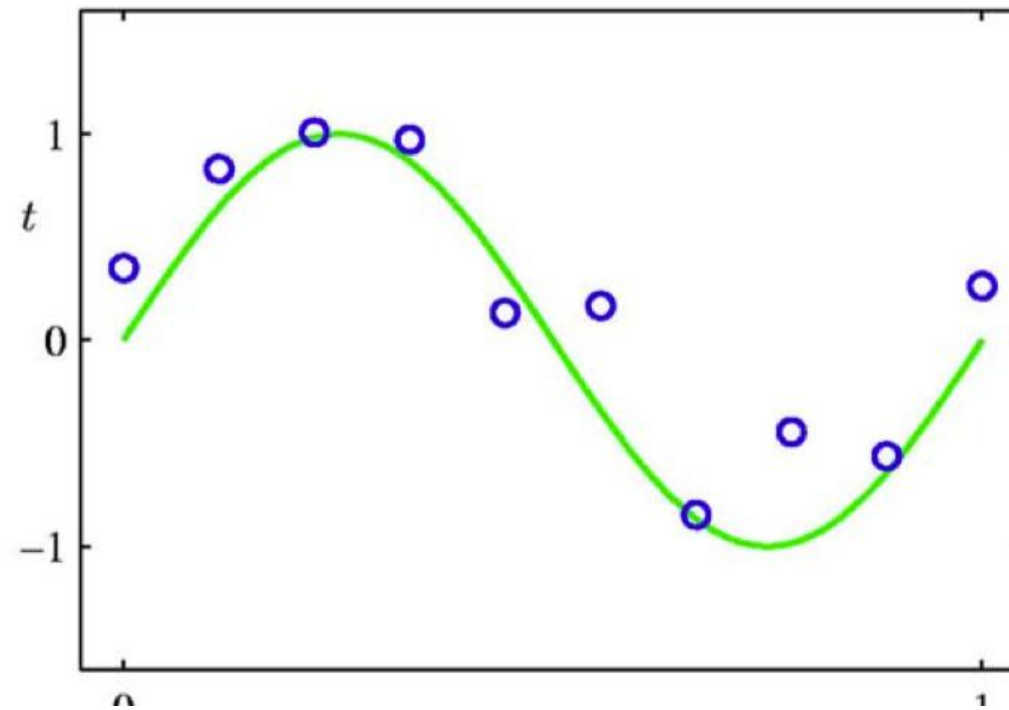
- Overfitting and regularized learning 
- Ridge regression
- Lasso regression
- Determining regularization strength

Regression: Recap



- Suppose we are given a training set of N observations
- Regression problem is to estimate $y(x)$ from this data

Regression: Recap



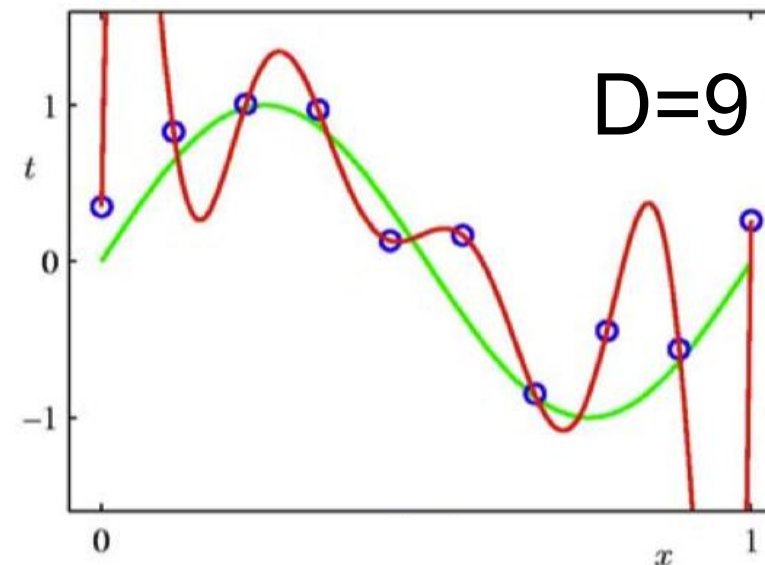
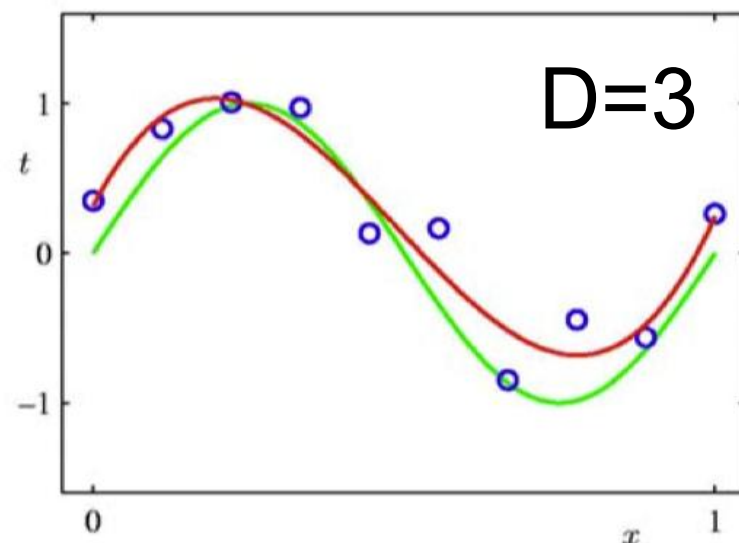
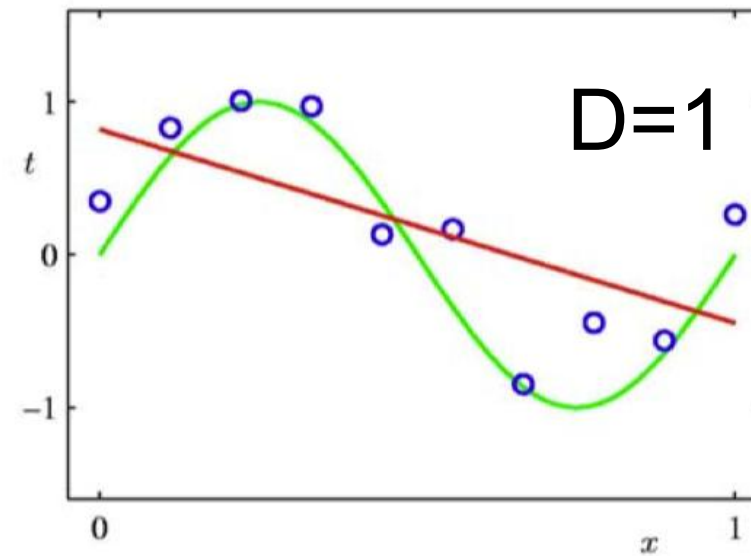
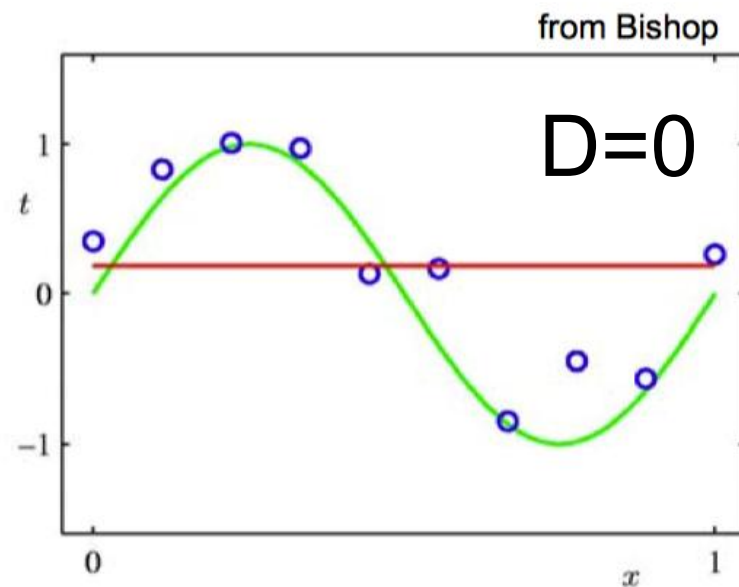
- Want to fit a polynomial regression model

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \dots + \theta_d x^d + \epsilon$$

- $z = \{1, x, x^2, \dots, x^d\} \in R^d$ and $\theta = (\theta_0, \theta_1, \theta_2, \dots, \theta_d)^T$

$$y = z\theta$$

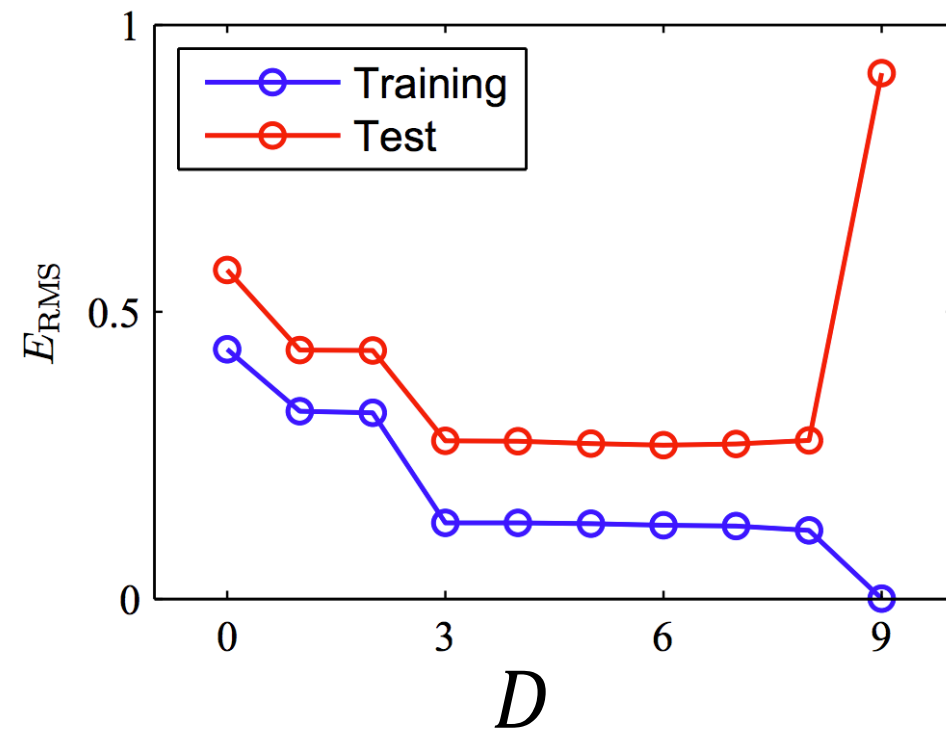
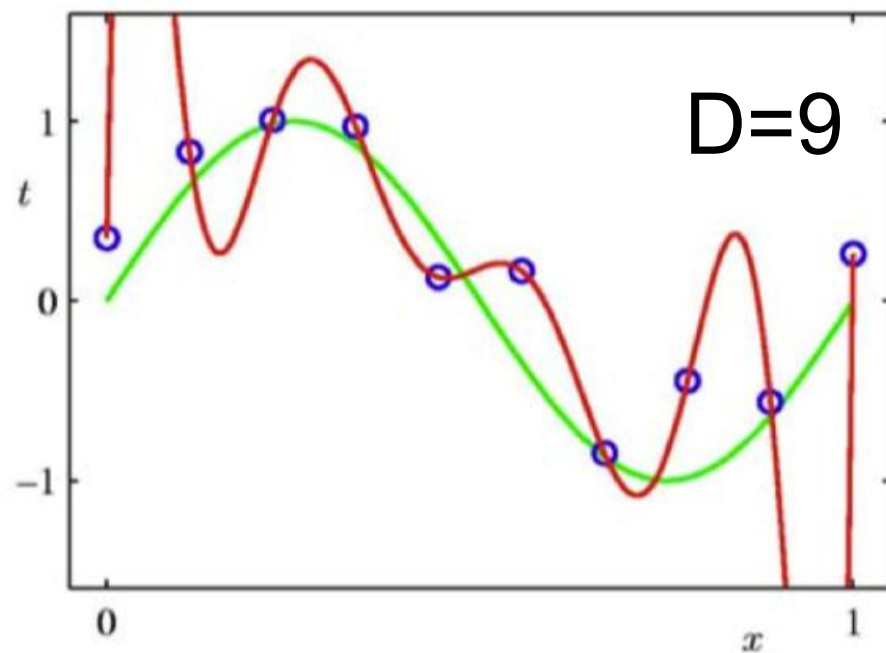
Which One is Better?



- Can we increase the maximal polynomial degree to very large, such that the curve passes through all training points?

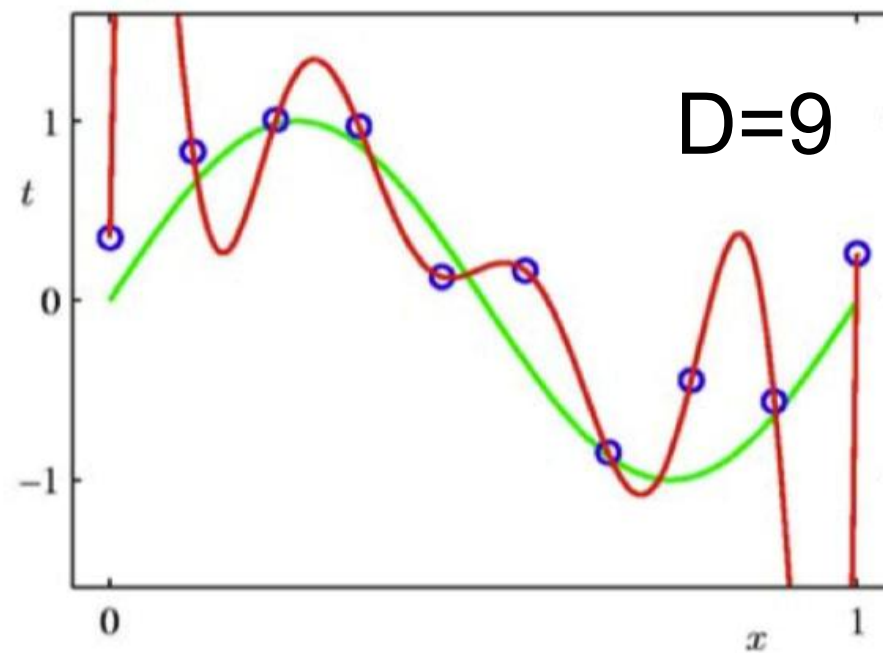
No, this can lead to **overfitting**!

The Overfitting Problem



- The training error is very low, but the error on test set is large.
- The model captures not only patterns but also noisy nuisances in the training data.

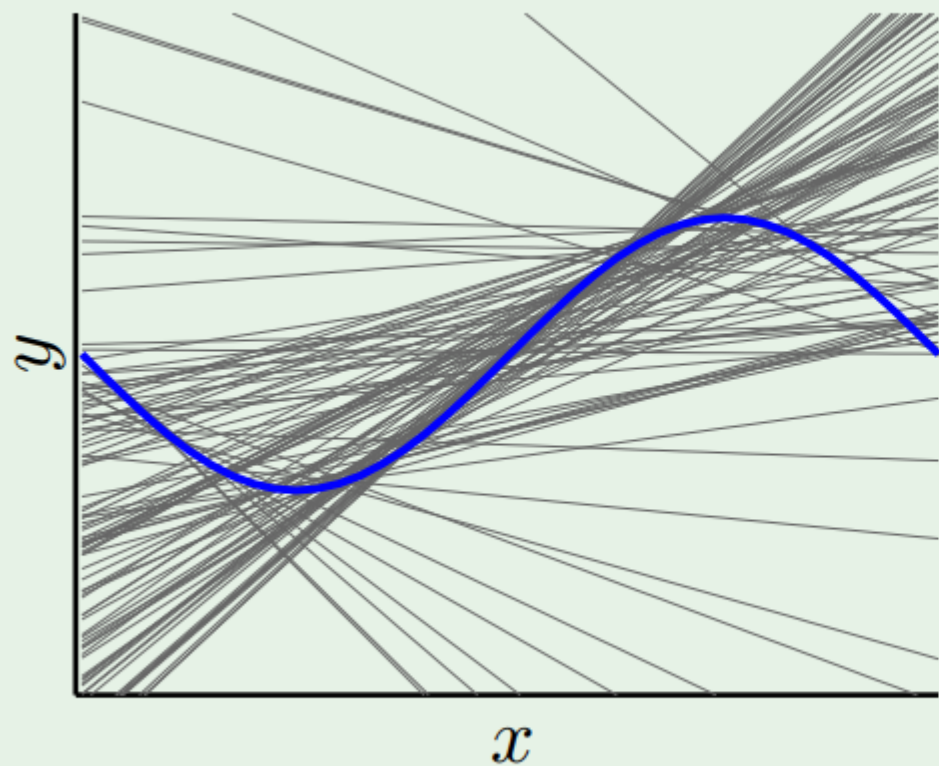
The Overfitting Problem



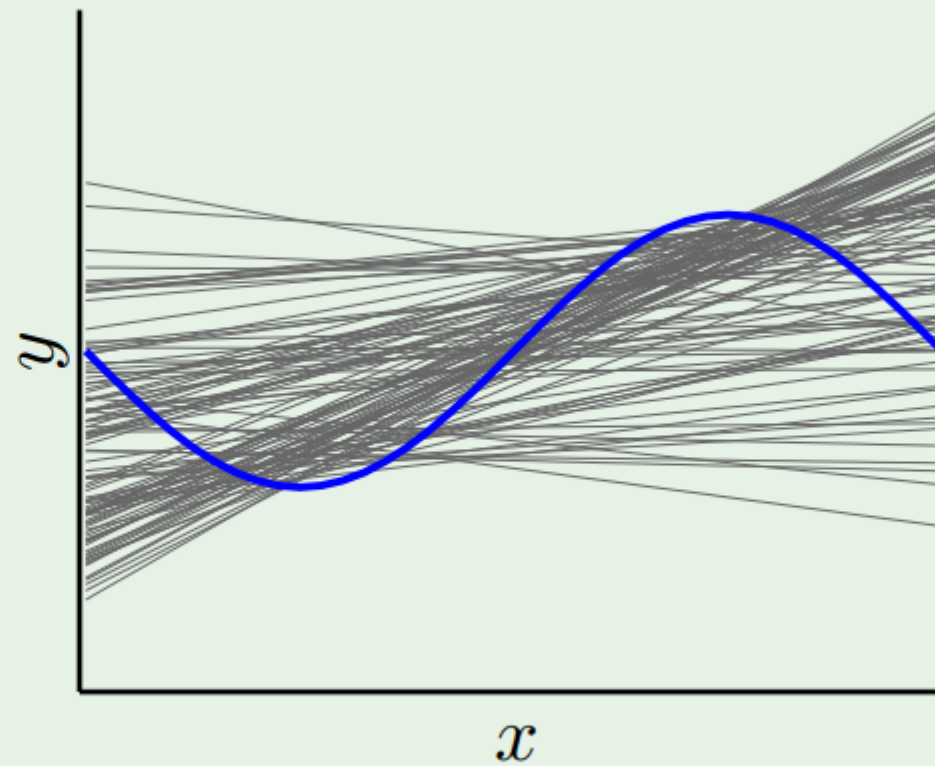
- In regression, overfitting is often associated with large Weights (**severe oscillation**)
- How can we address overfitting?

Regularization

(smart way to cure overfitting disease)



without regularization



with regularization

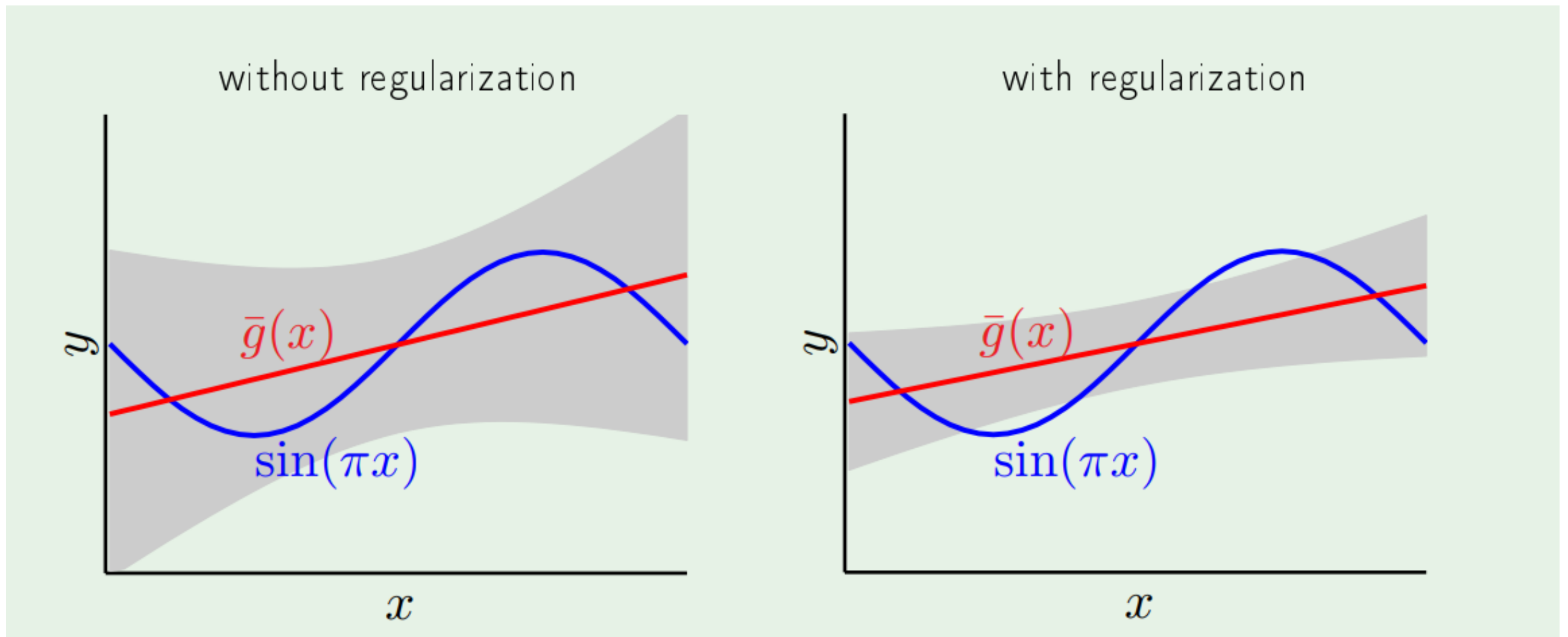
Put a brake on fitting



Fit a linear line on sinusoidal with just two data points

Who is the winner?

$\bar{g}(x)$: average over all lines



bias=0.21; var=1.69

bias=0.23; var=0.33

Polynomial Model

Want to fit a polynomial regression model

$$y = \theta_0 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_d x^d + \epsilon$$

Let's rewrite it as:

$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_d + \epsilon = \mathbf{z}\boldsymbol{\theta}$$

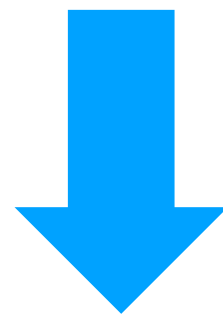
Regularizing is just constraining the weights (θ)

For example: let's do a **hard** constraining

$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_d$$

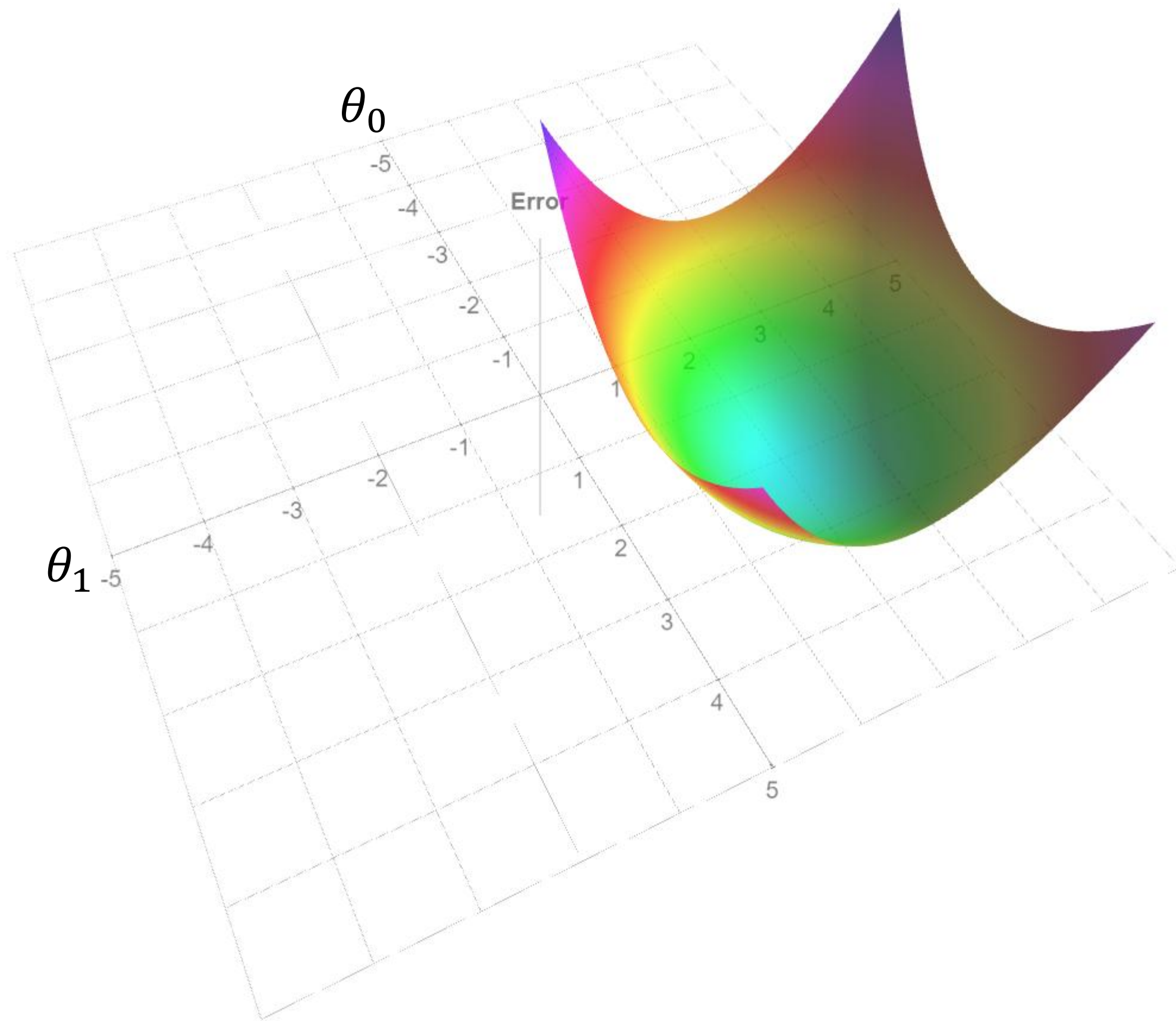
subject to

$$\theta_d = 0 \text{ for } d > 2$$

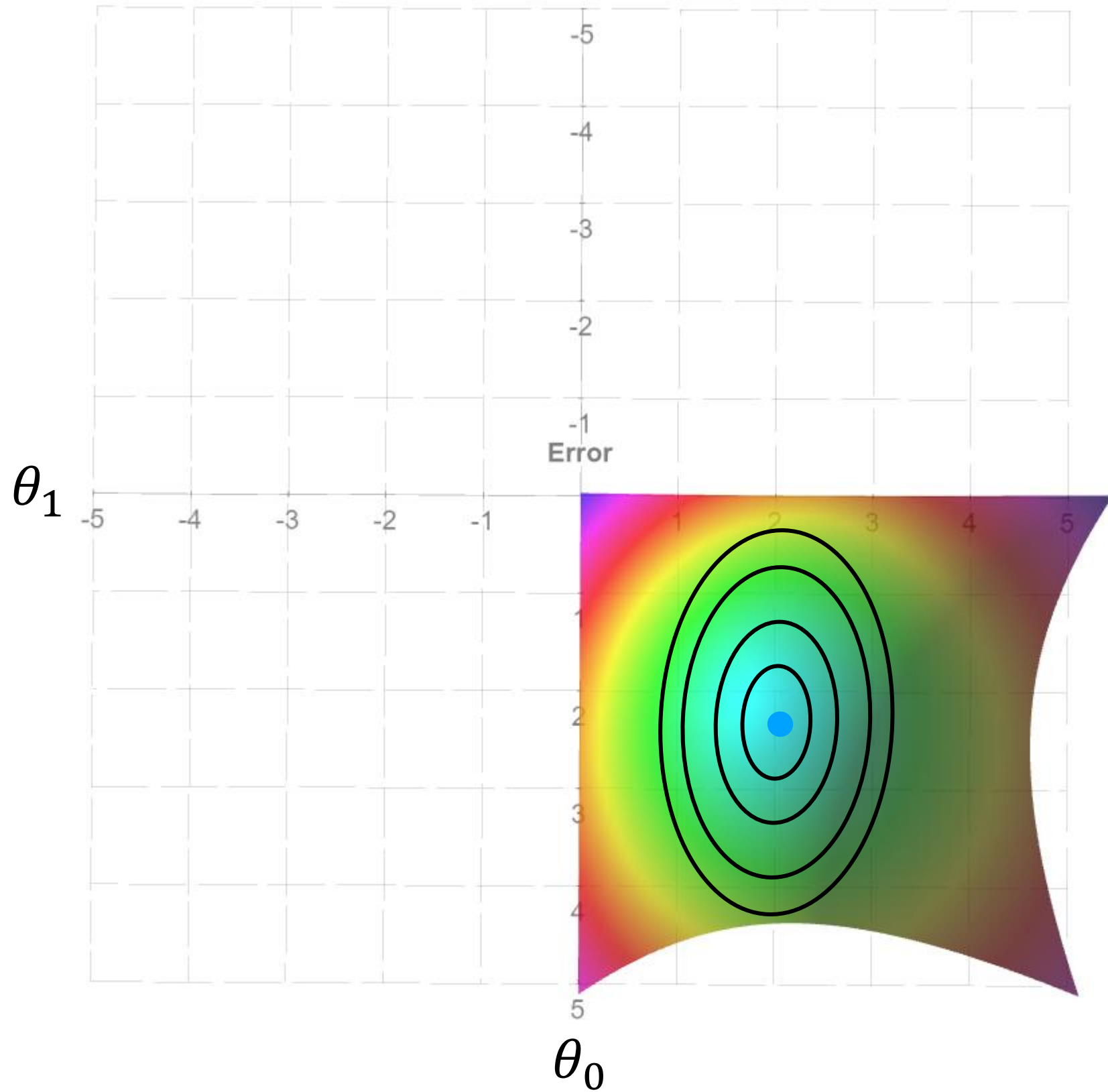


$$y = \theta_0 + \theta_1 z_1 + \theta_2 z_2 + 0 + \cdots + 0$$

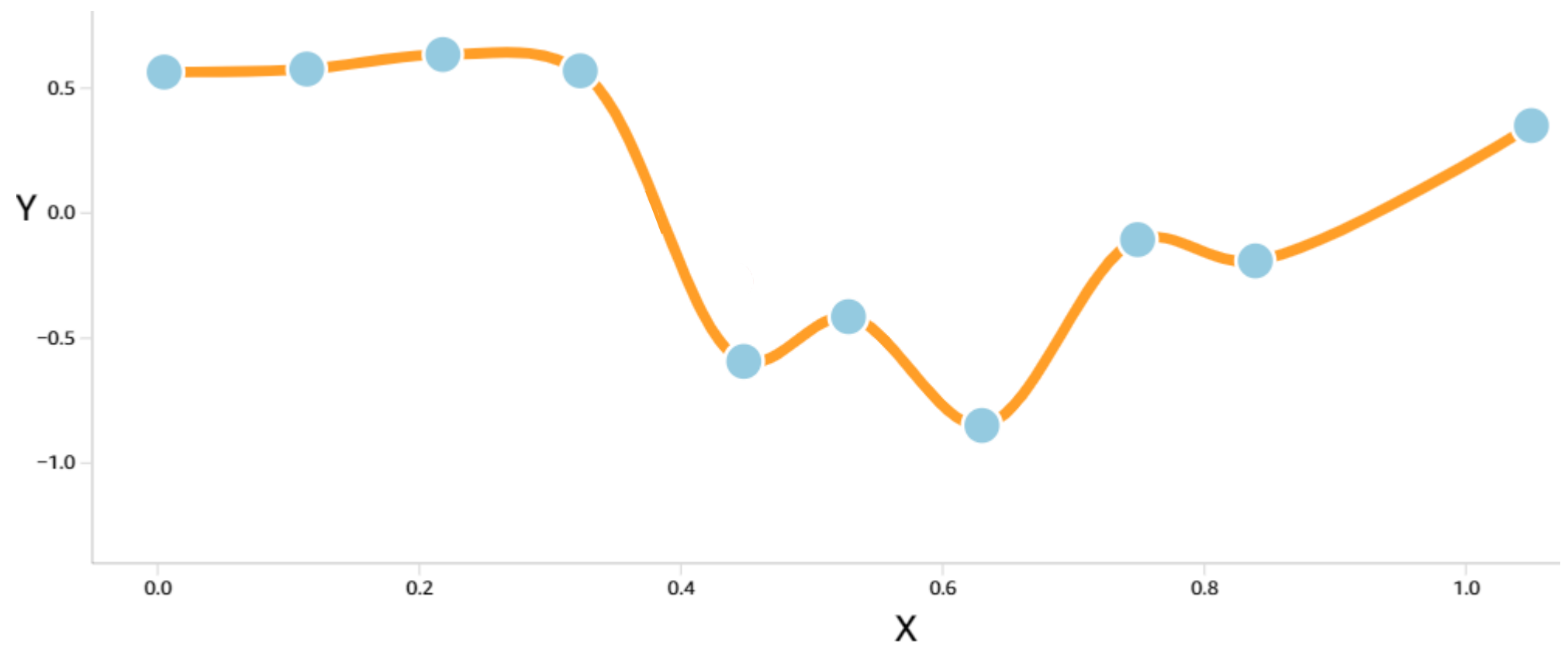
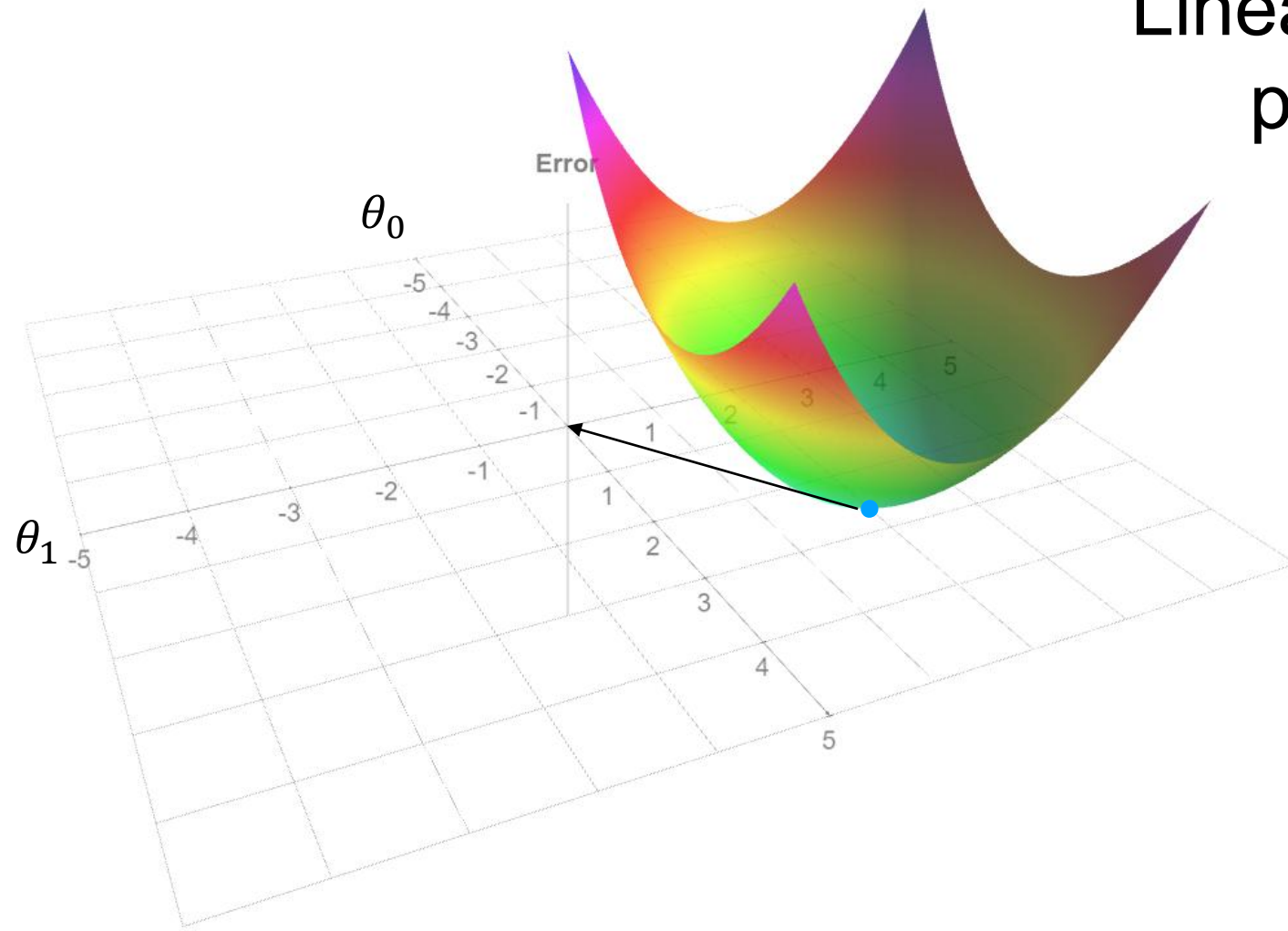
$$E(\theta) = \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2$$

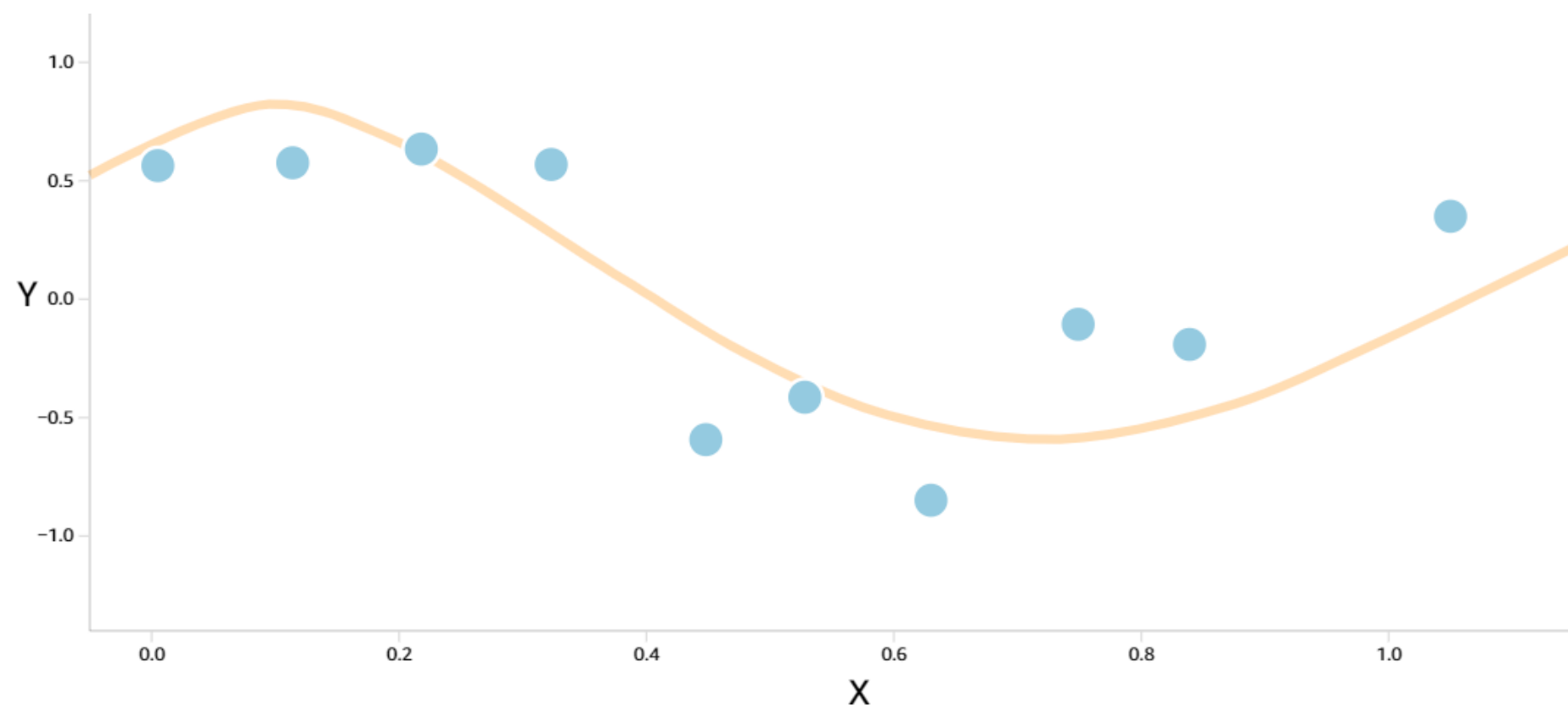
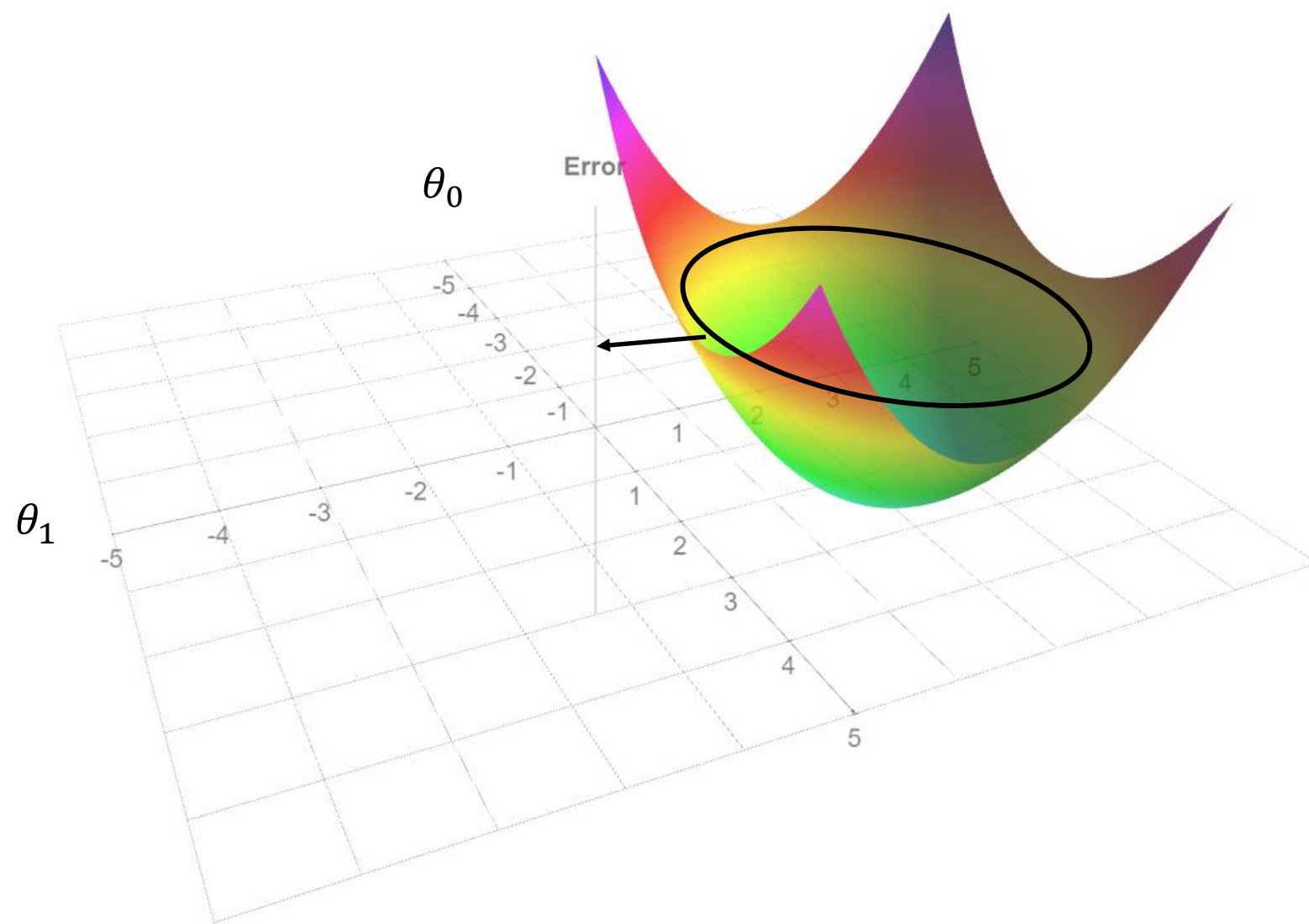


Project the same graph on x-y using contour plot



Linear regression with a very high polynomial degree solution



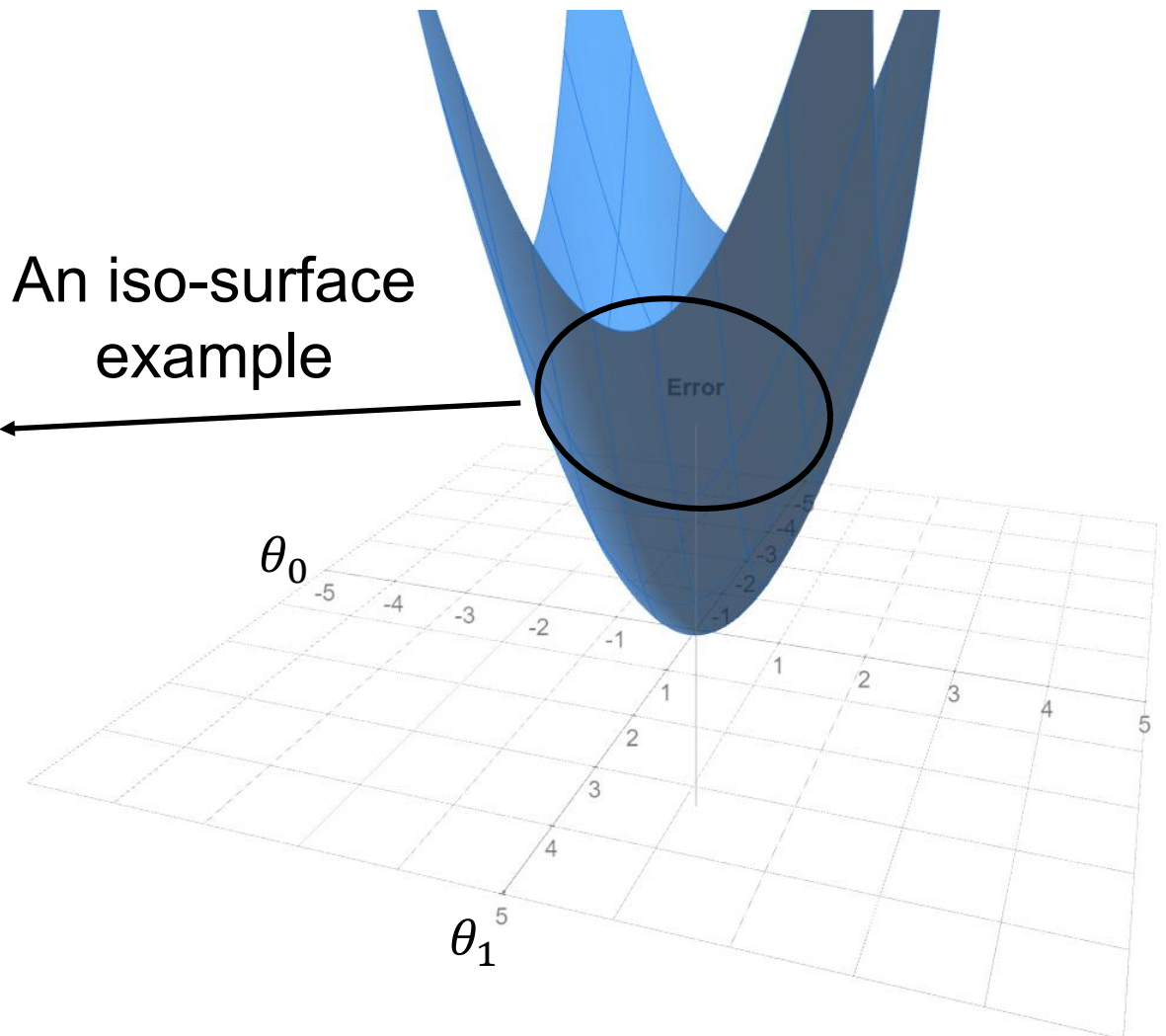


How can we get an optimal solution with a positive error for a model that overfits?

We need to introduce a constraint

$$g(\theta) = \theta_0^2 + \theta_1^2 = \theta^T \theta = C$$

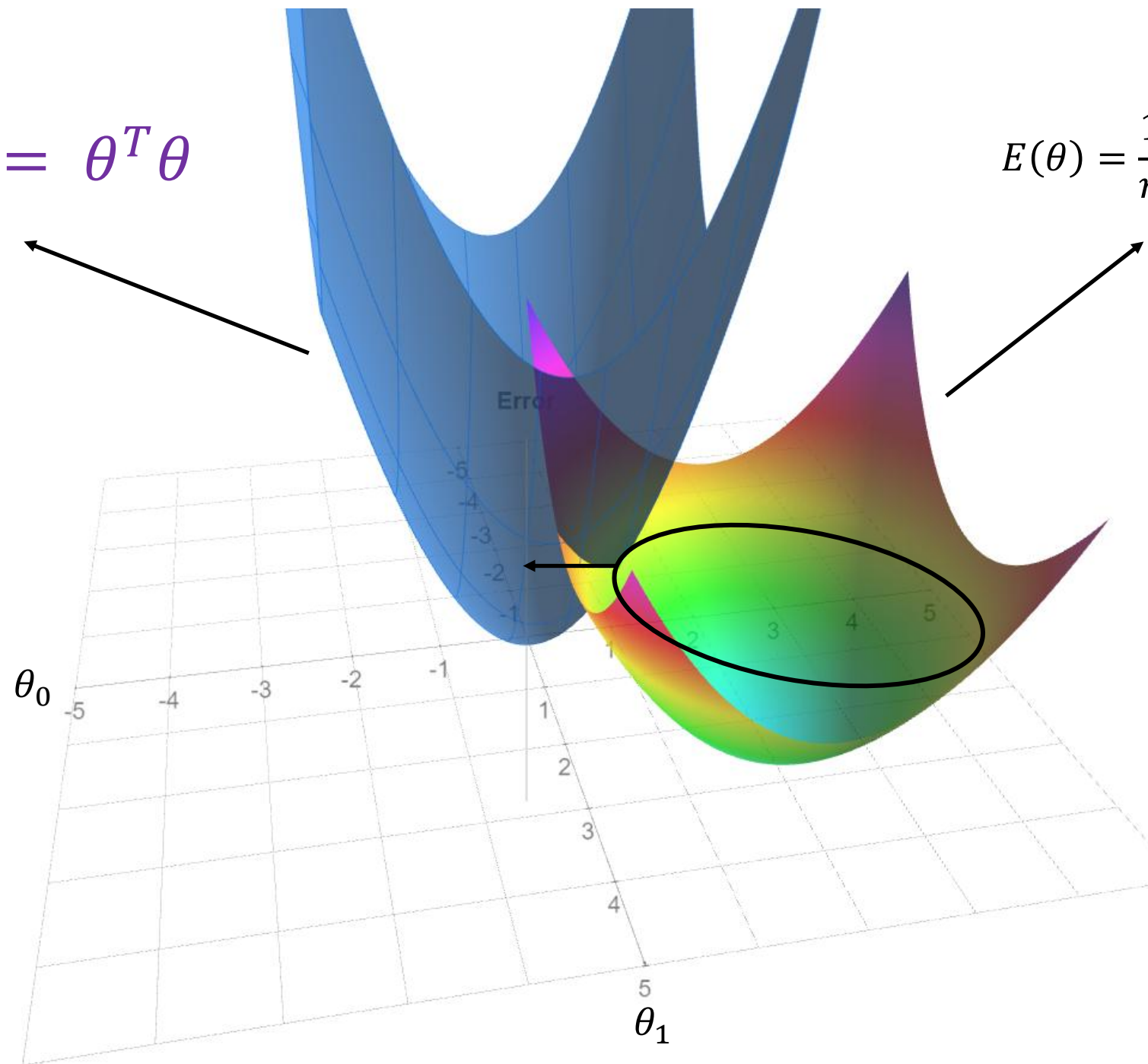
An iso-surface
example



Error function together with a
new introduced constraint

$$g(\theta) = \theta_0^2 + \theta_1^2 = \theta^T \theta$$

$$E(\theta) = \frac{1}{n} \sum_{i=1}^n (y^i - z_i \theta)^2$$



Let's define the Lagrange function

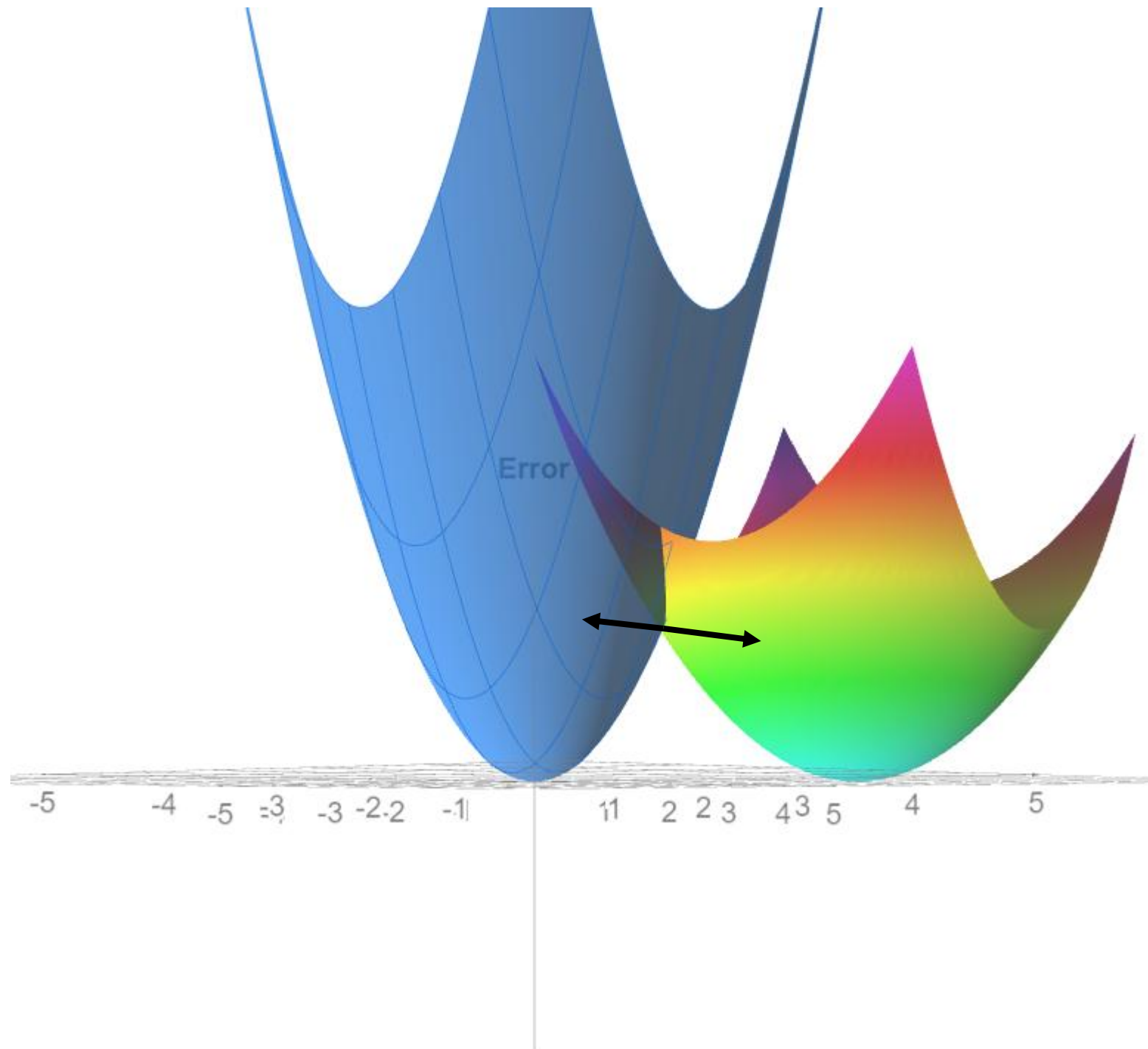
$$L(\theta, \lambda) = E(\theta) + \lambda g(\theta)$$

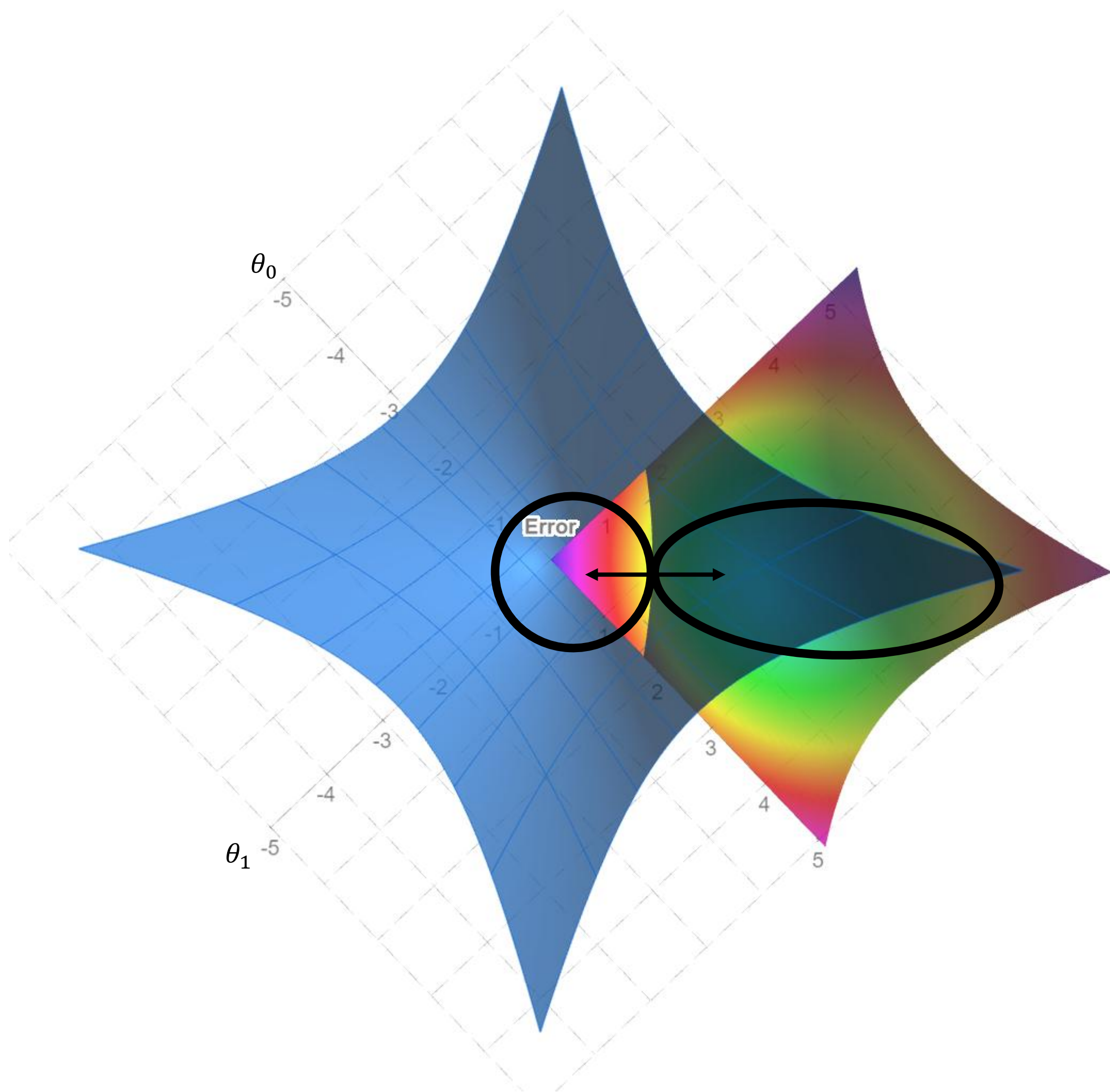
$$L(\theta, \lambda) = E(\theta) + \lambda \theta^T \theta$$

$$\nabla L(\theta, \lambda) = 0 \qquad \nabla [E(\theta) + \lambda \theta^T \theta] = 0$$

$$\nabla [E(\theta)] + \lambda \nabla [\theta^T \theta] = 0$$

How to enforce the gradient of Lagrange function to be zero





Let's calculate the gradients

Gradient of constraint $g(\theta)$

$$\nabla[\theta^T \theta] = 2\theta$$

$$\nabla[E(\theta)] + \lambda \nabla[\theta^T \theta] = 0$$

$$\nabla[E(\theta)] = -\lambda \nabla[\theta^T \theta]$$

$$\nabla E(\theta) = -2\lambda\theta$$

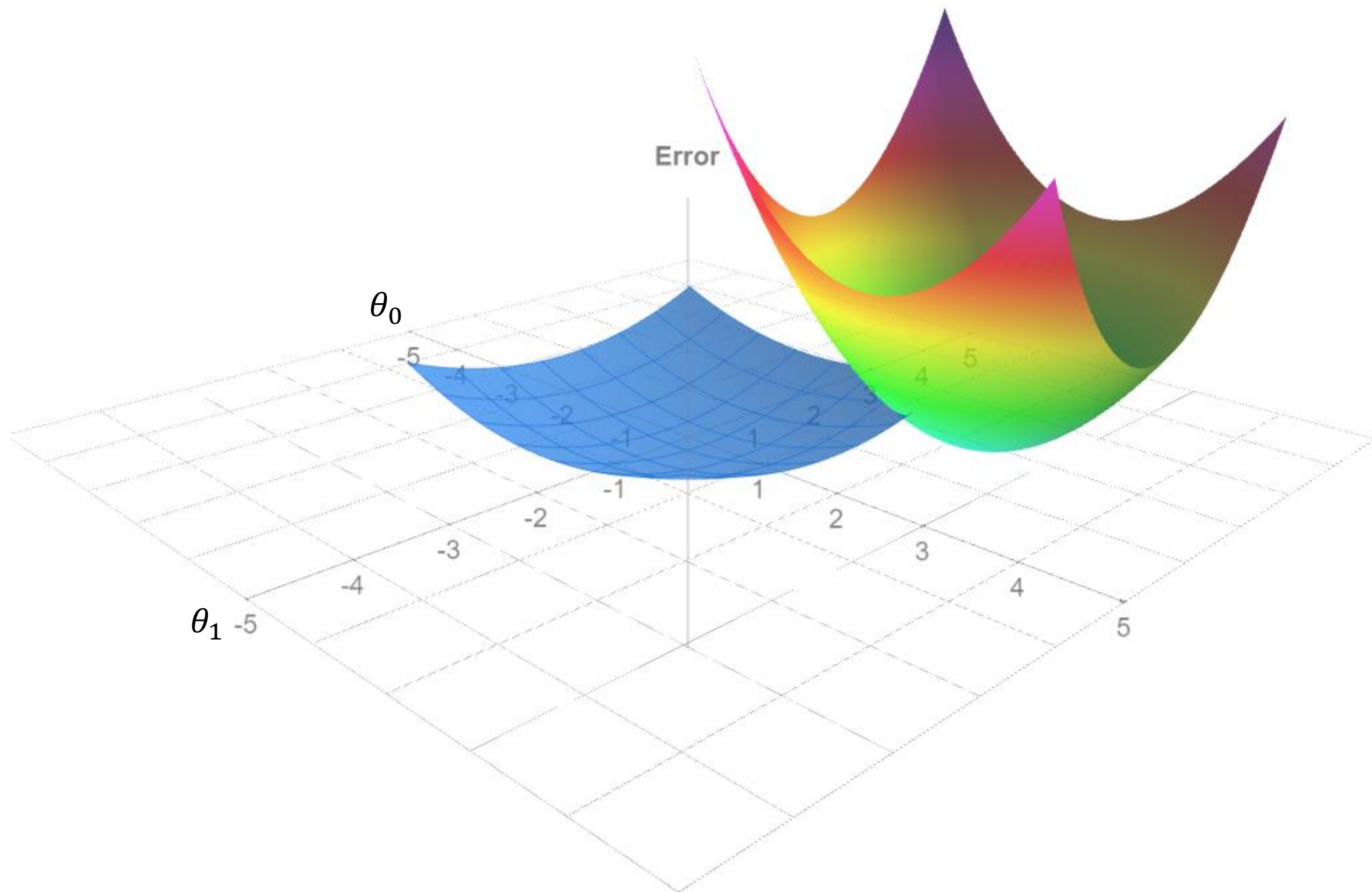
$$\nabla E(\theta) + 2\lambda\theta = 0$$

Let's do integration

$$E(\theta) + \lambda\theta^T \theta$$

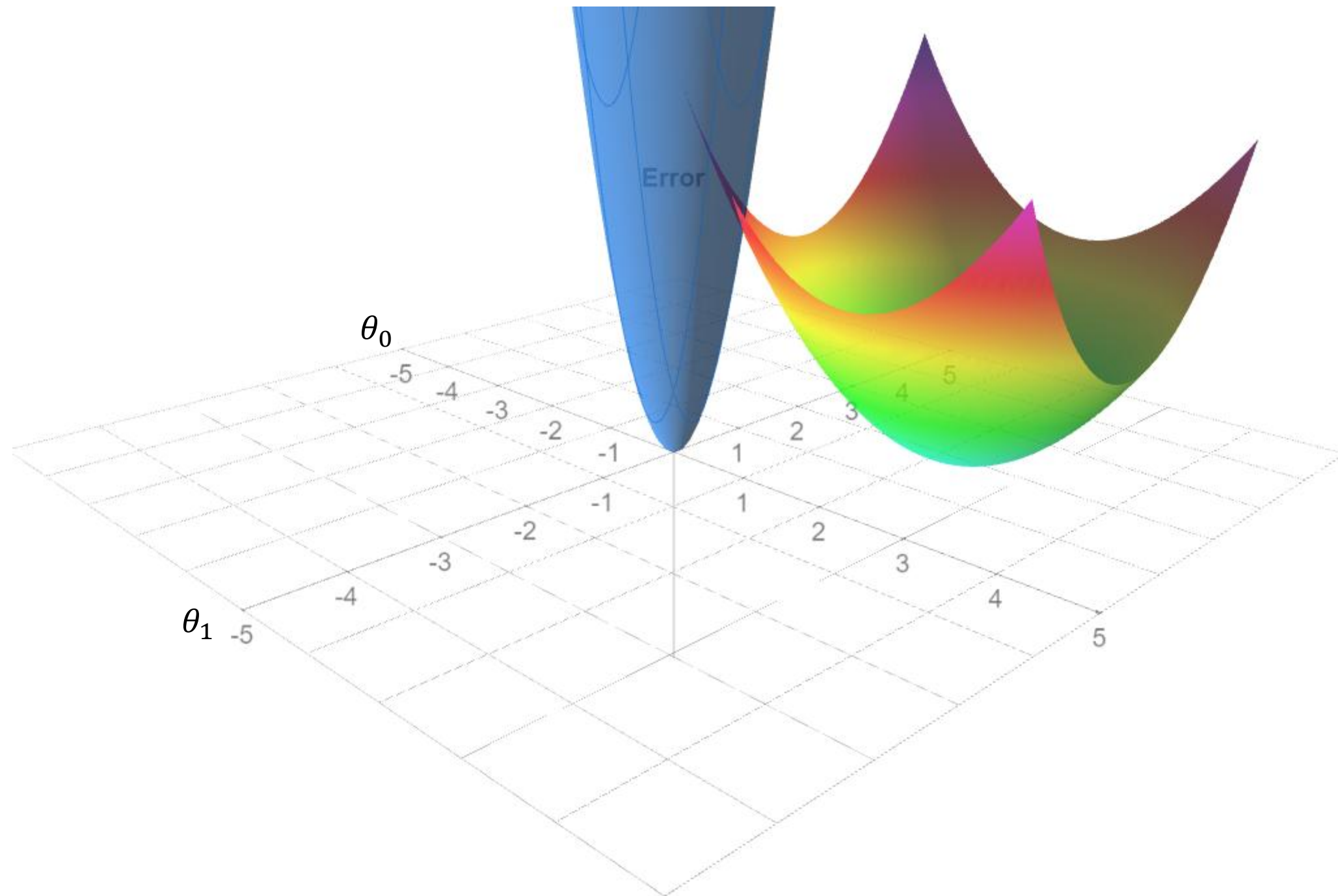
The effect of low Lambda

$$E(\theta) + \frac{\lambda}{N} \theta^T \theta$$



The effect of high Lambda

$$E(\theta) + \frac{\lambda}{N} \theta^T \theta$$



Regularized Learning

Minimize $E(\theta) + \lambda \theta^T \theta$

Now we know Why this term
leads to the regularization of
parameters



Regularized Error $\tilde{E}(\theta)$

$$= \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2 + \frac{\lambda}{2N} \|\theta\|_2^2$$

L2 Regularization term

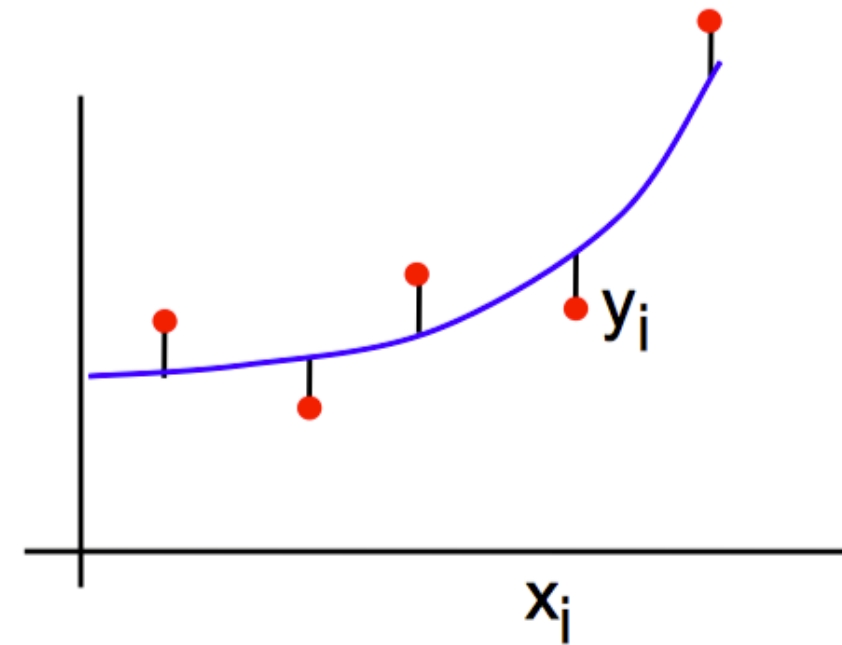


Outline

- Overfitting and regularized learning
- Ridge regression 
- Lasso regression
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Ridge Regression

$$\begin{aligned}\tilde{E}(\theta) \\ &= \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2 + \lambda \|\theta\|_2^2\end{aligned}$$



$$\theta_0 + \theta_1 z_1 + \theta_2 z_2 + \cdots + \theta_d z_d + \epsilon = \mathbf{z} \boldsymbol{\theta}$$

General form

$$\tilde{E}(\theta) = \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2 + \lambda \|\theta\|_2^2$$

Matrix form

$$\tilde{E}(\theta) = \frac{1}{N} (y - z\theta)^T (y - z\theta) + \lambda \|\theta\|_2^2$$

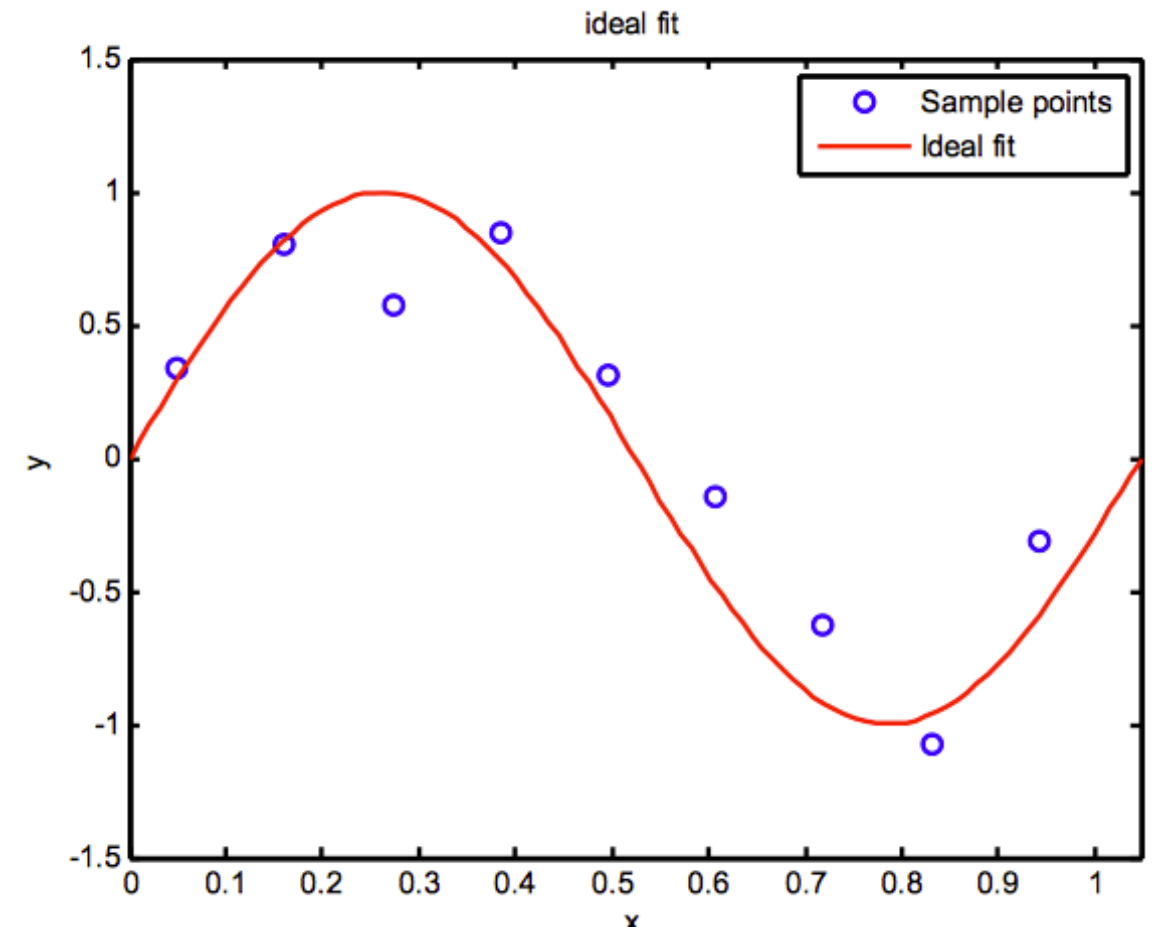
$$\frac{\partial \tilde{E}(\theta)}{\partial \theta} = -z^T (y - z\theta) + \lambda \theta$$

$$(z^T z + \lambda I) \theta = z^T y$$

$$\theta = (z^T z + \lambda I)^{-1} z^T y$$

Ridge Regression Example

- The red curve is the true function (which is not a polynomial)
- The data points are samples from the curve with added noise in y.
- There is a choice in both the degree, D , of the basis functions used, and in the strength of the regularization

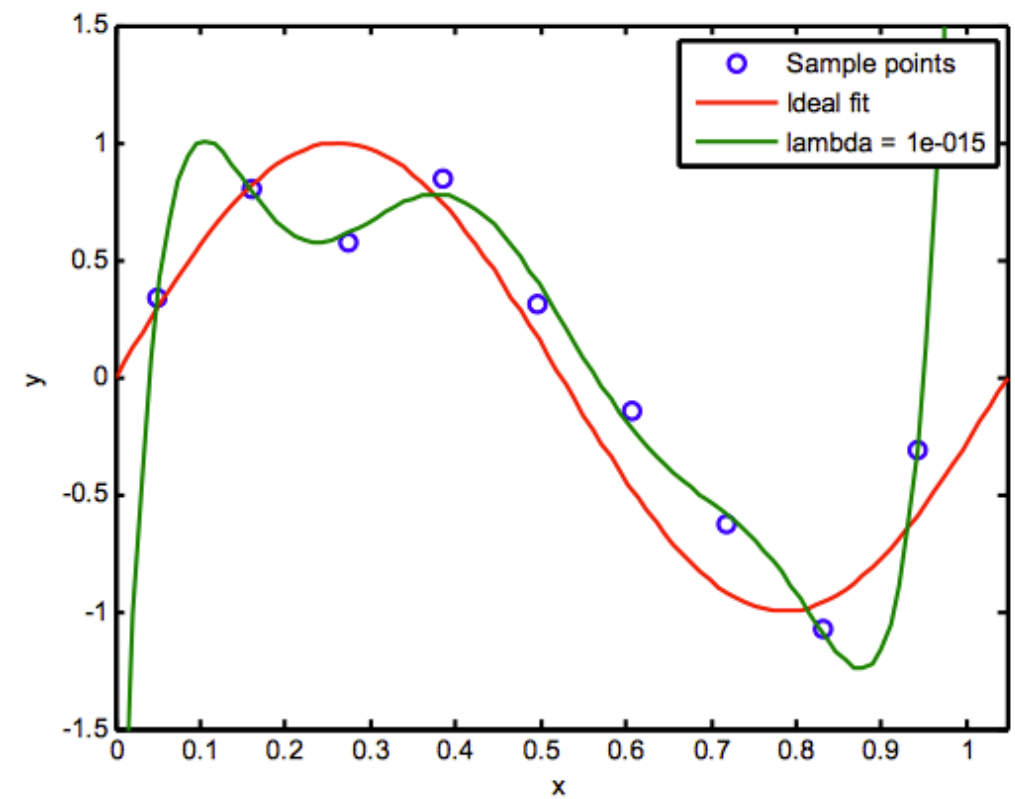
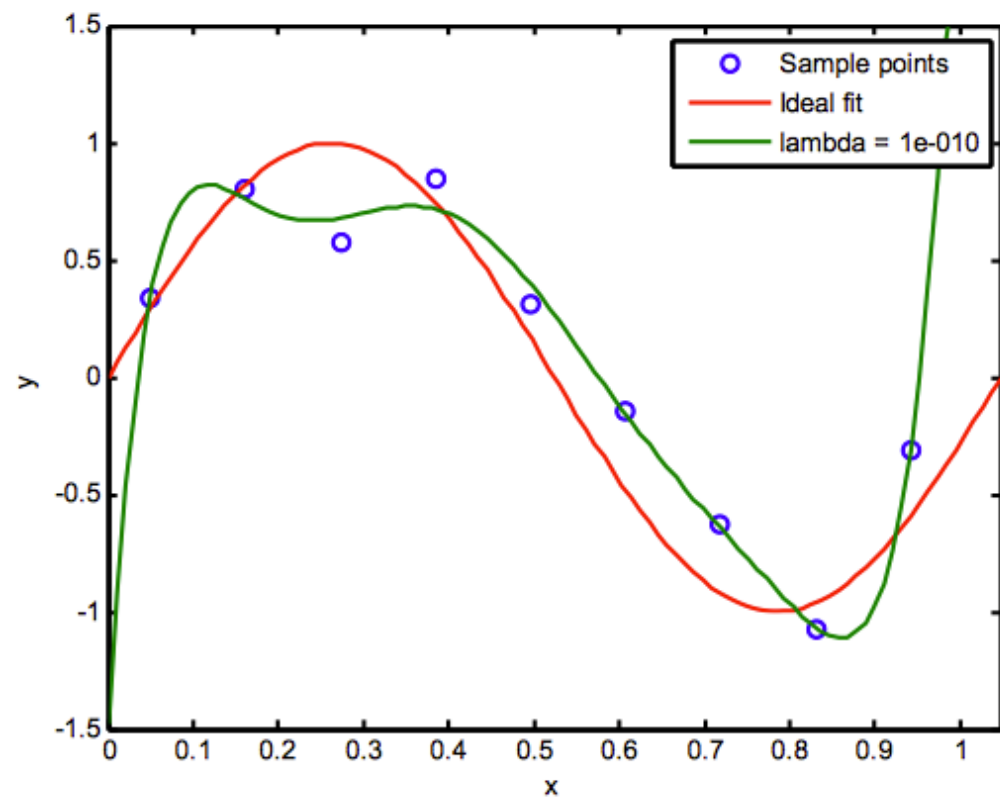
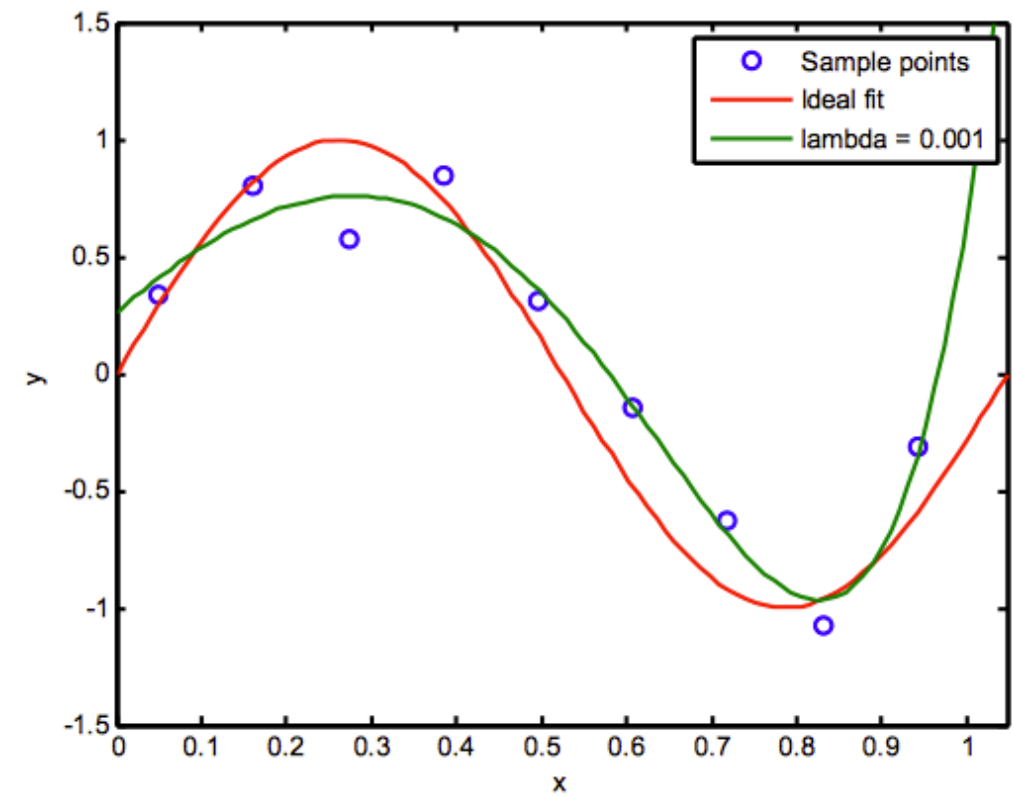
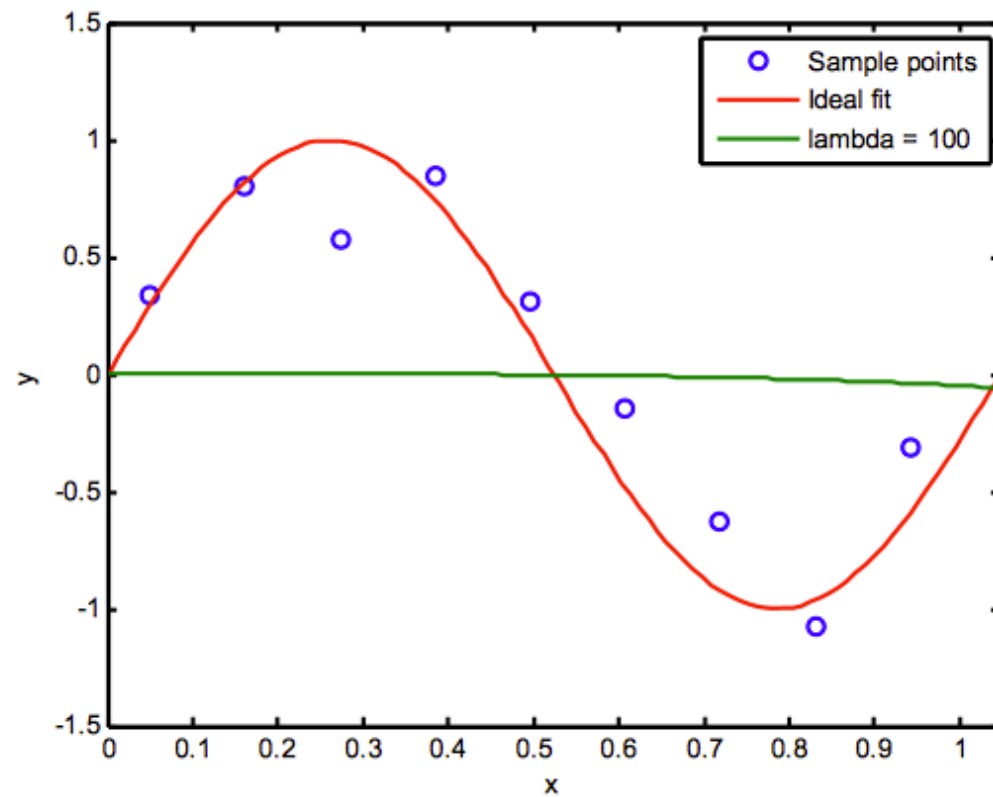


$$f(x, \theta) = z\theta$$

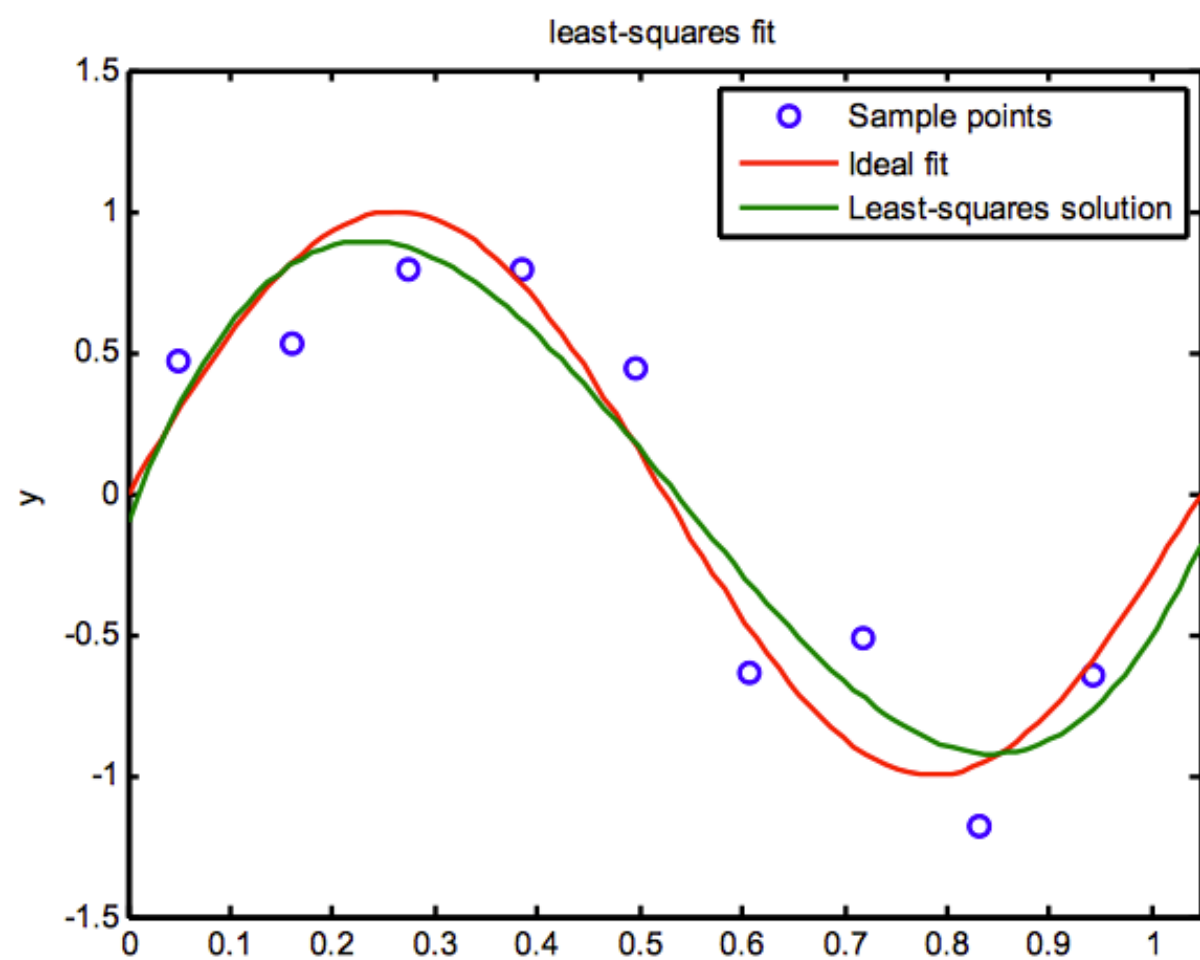
$$z: x \rightarrow z$$

$$\tilde{E}(\theta) = \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}}\theta)^2 + \lambda \|\theta\|_2^2 \quad \theta \in \mathbb{R}^{D+1}$$

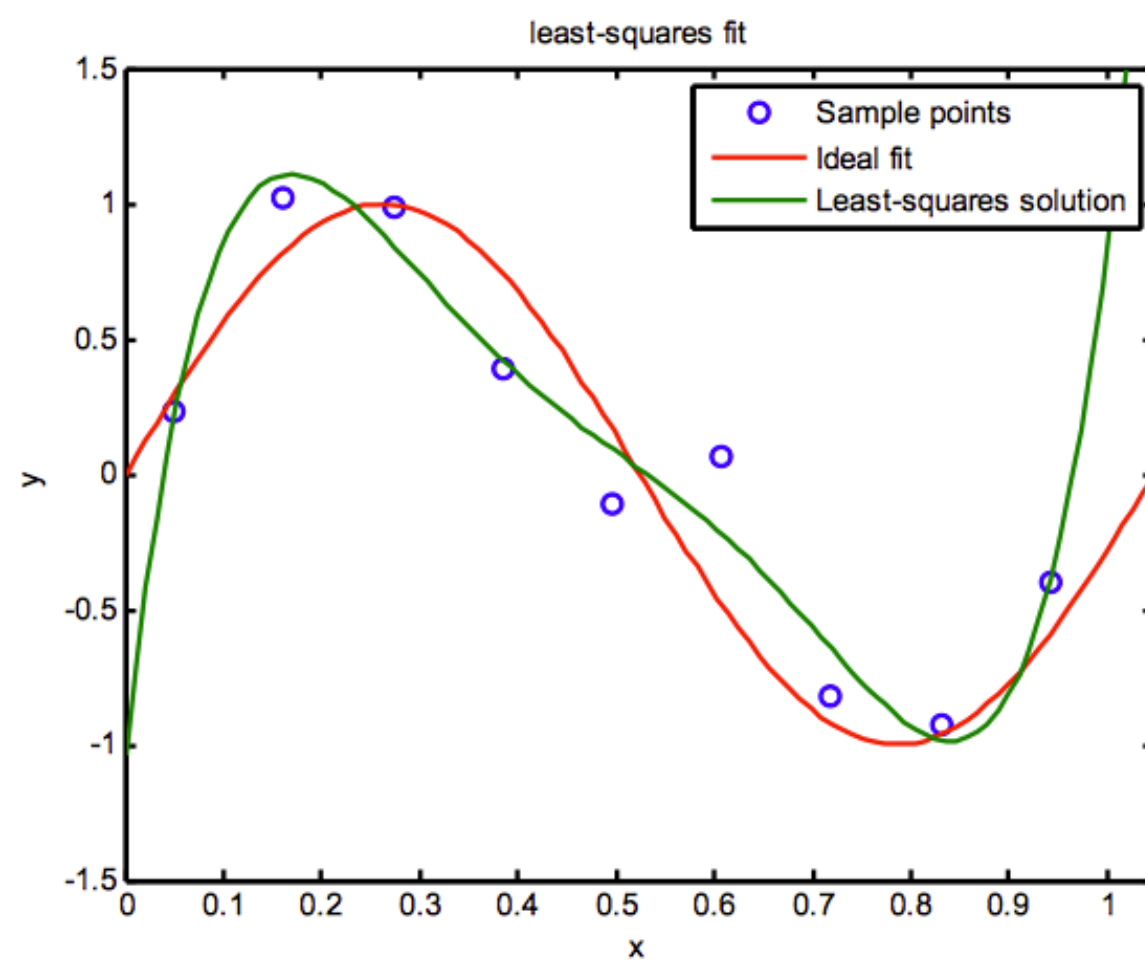
$N = 9$ samples, $D = 7$



$D = 3$



$D = 5$



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Regularized Regression

$$\tilde{E}(\theta) = \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2 + \lambda \|\theta\|_2^2$$

Squared loss\Error

$$\frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2$$

L2 Regularizer

$$\lambda \|\theta\|_2^2$$

Now let's look at another regularization choice.

The Lasso Regularization (L1 norm) and sparsity

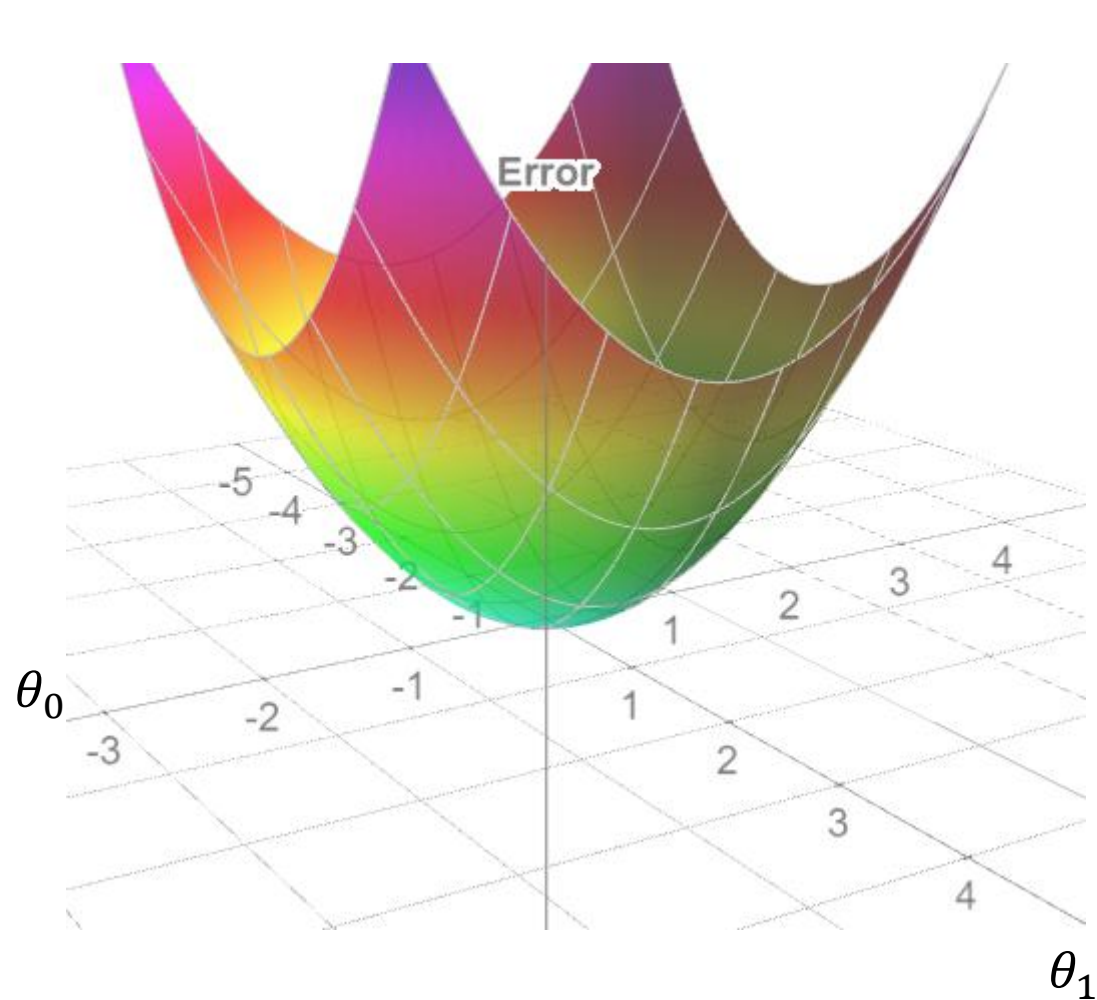
Lasso = **L**east Absolute **S**hrinkage and **S**election **O**perator

$$\tilde{E}(\theta) = \frac{1}{N} \sum_{i=1}^n (y^{\{i\}} - z^{\{i\}} \theta)^2 + \lambda \|\theta\|_1$$

L1 norm induces sparsity. This means that some of the weights become zero, and the feature contribution will be completely removed. L1 Regularizer could be used for feature selection

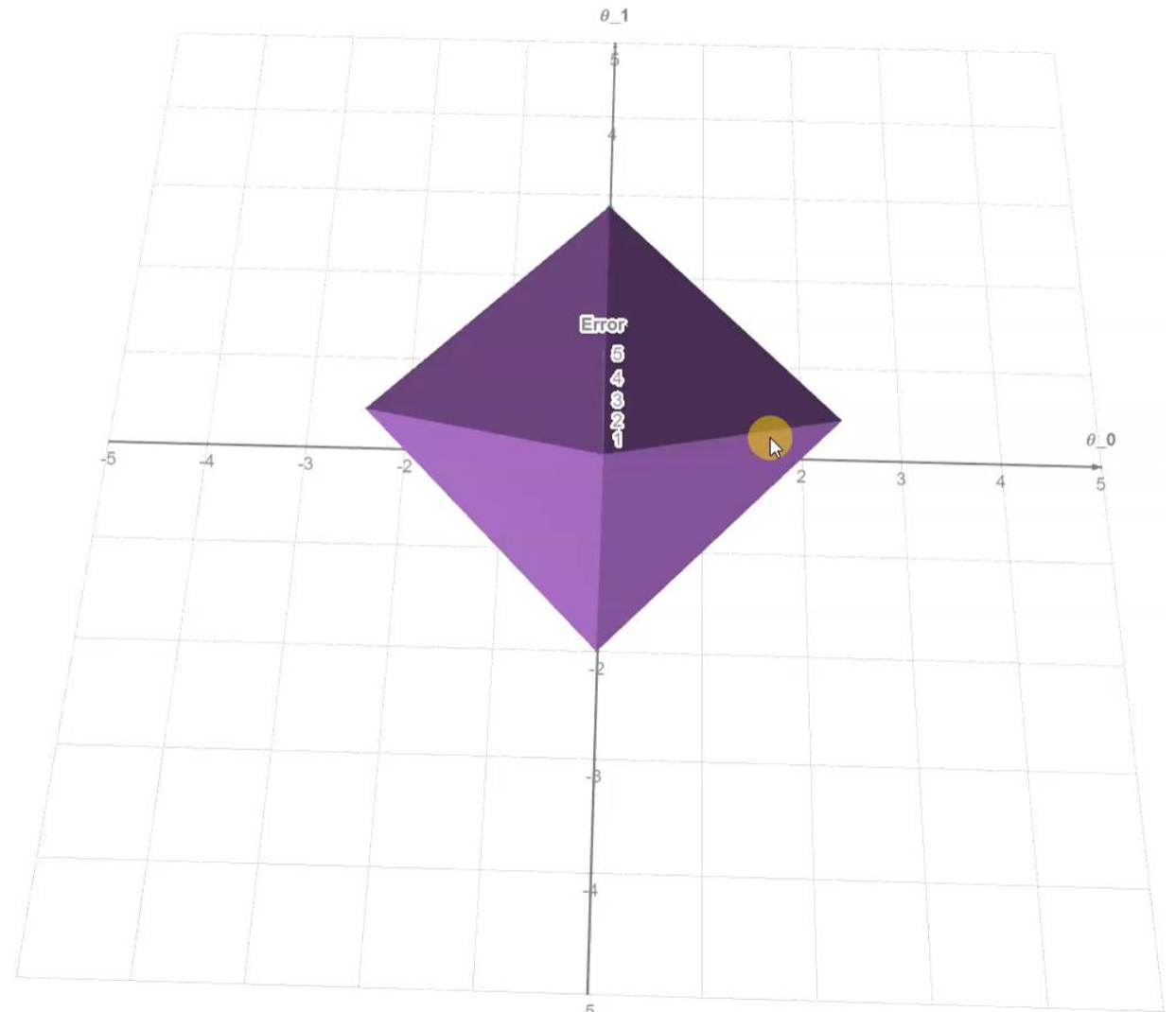
Ridge Regularizer

$$g(\theta) = \theta_0^2 + \theta_1^2 = \theta^T \theta$$



Lasso Regularizer

$$g(\theta) = \theta_0 + \theta_1 = \theta$$



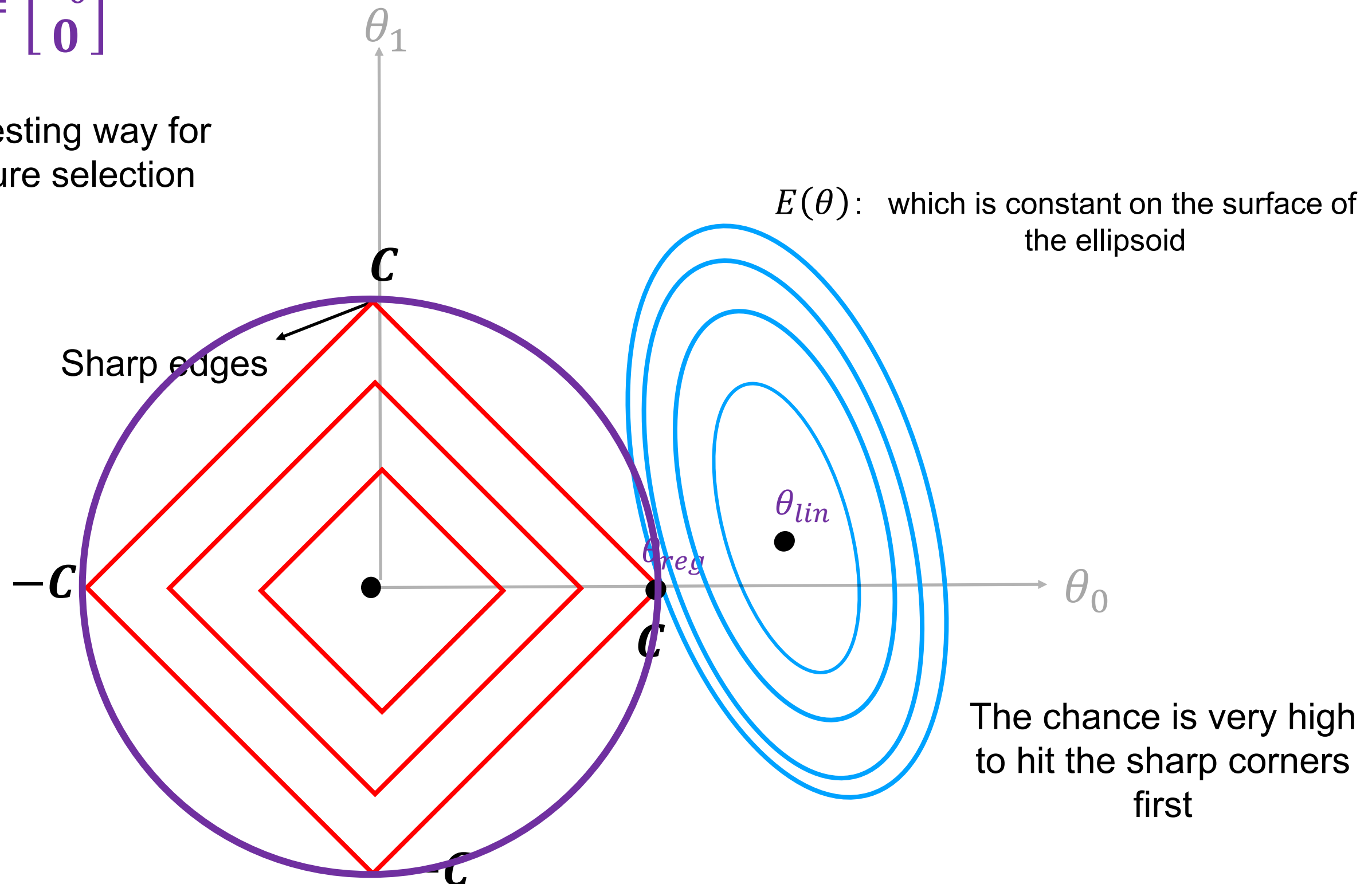
[Animation](#)

Let's say we have two parameters (θ_0 and θ_1)

$$\text{Min } E(\theta) = \frac{1}{N} (z\theta - y)^T (z\theta - y) + \lambda \|\theta\|_1$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \end{bmatrix}$$

Interesting way for
feature selection



Ridge versus Lasso

Ridge

$$\tilde{E}(\theta) = \frac{1}{N} (y - z\theta)^T (y - z\theta) + \lambda \|\theta\|_2^2$$

It is a convex model

Both mean squared error
and L2 regularizer are
differentiable.

We can get a closed form
solution

Lasso

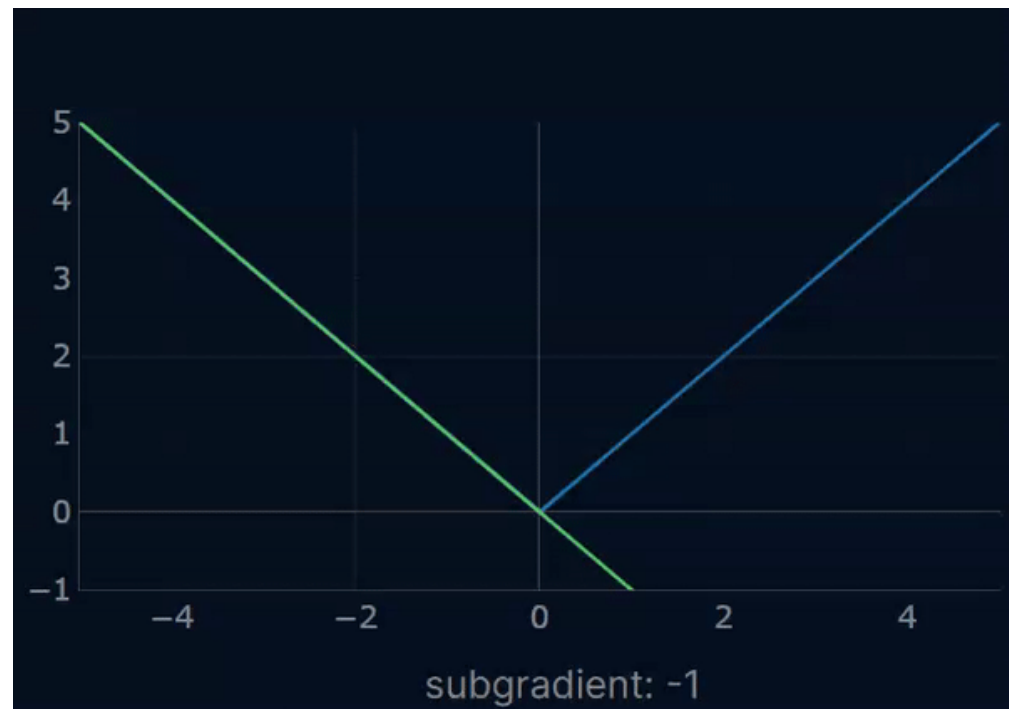
$$\tilde{E}(\theta) = \frac{1}{N} (y - z\theta)^T (y - z\theta) + \lambda \|\theta\|_1$$

It is a convex model

L1 regularizer is NOT
differentiable.

We can **NOT** get a closed
form solution

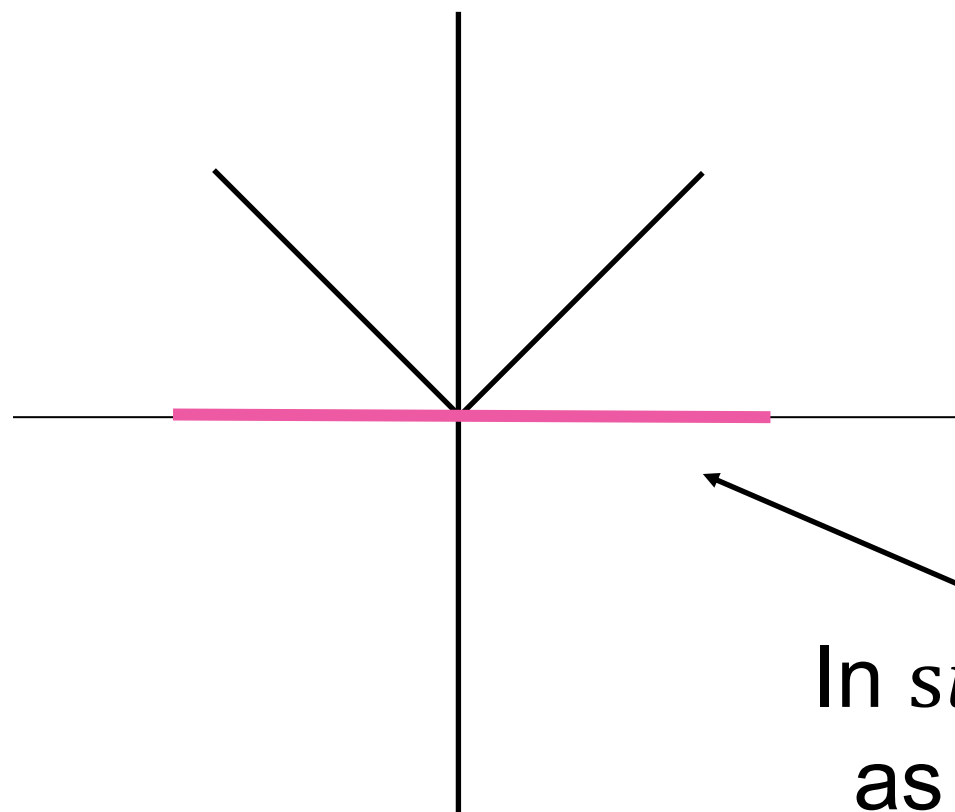
Sub-gradient Descend in Lasso



$$\tilde{E}(\theta) = \frac{1}{N} (y - z\theta)^T (y - z\theta) + \lambda \|\theta\|_1$$

$$\frac{\partial \tilde{E}(\theta)}{\partial \theta} = -z^T (y - z\theta) + \frac{\partial (\lambda \|\theta\|_1)}{\partial \theta}$$

Using Sub-gradient



$$\frac{\partial \tilde{E}(\theta)}{\partial \theta} = -z^T (y - z\theta) + \lambda \text{sign}(\theta)$$

In *sign* function, we use this sub-gradient line as our under-estimator (below our function)

Outline

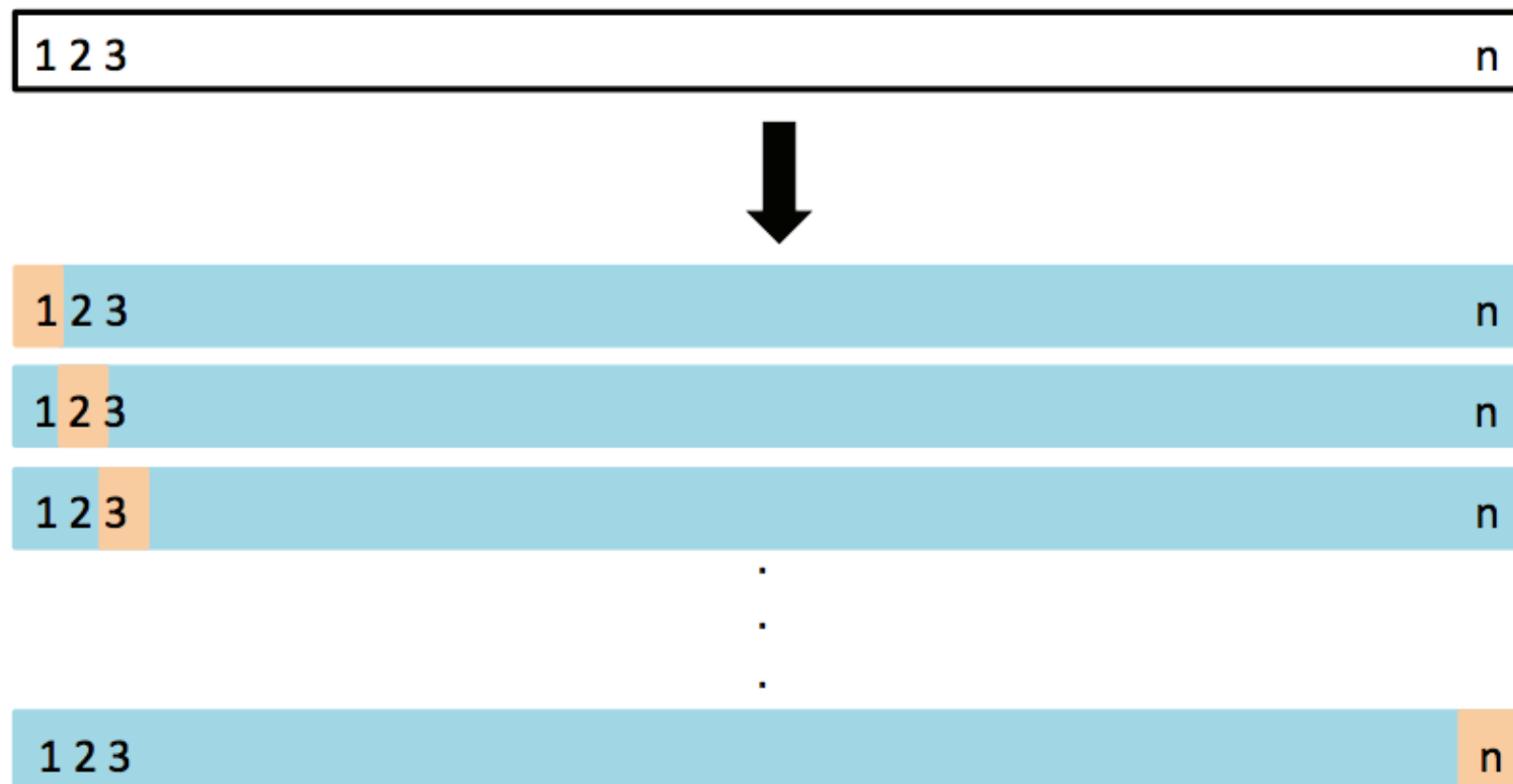
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Leave-One-Out Cross Validation

For every $i = 1, \dots, n$:

- ▶ train the model on every point except i ,
- ▶ compute the test error on the held out point.

Average the test errors. $CV_{(n)} = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i^{(-i)})^2$



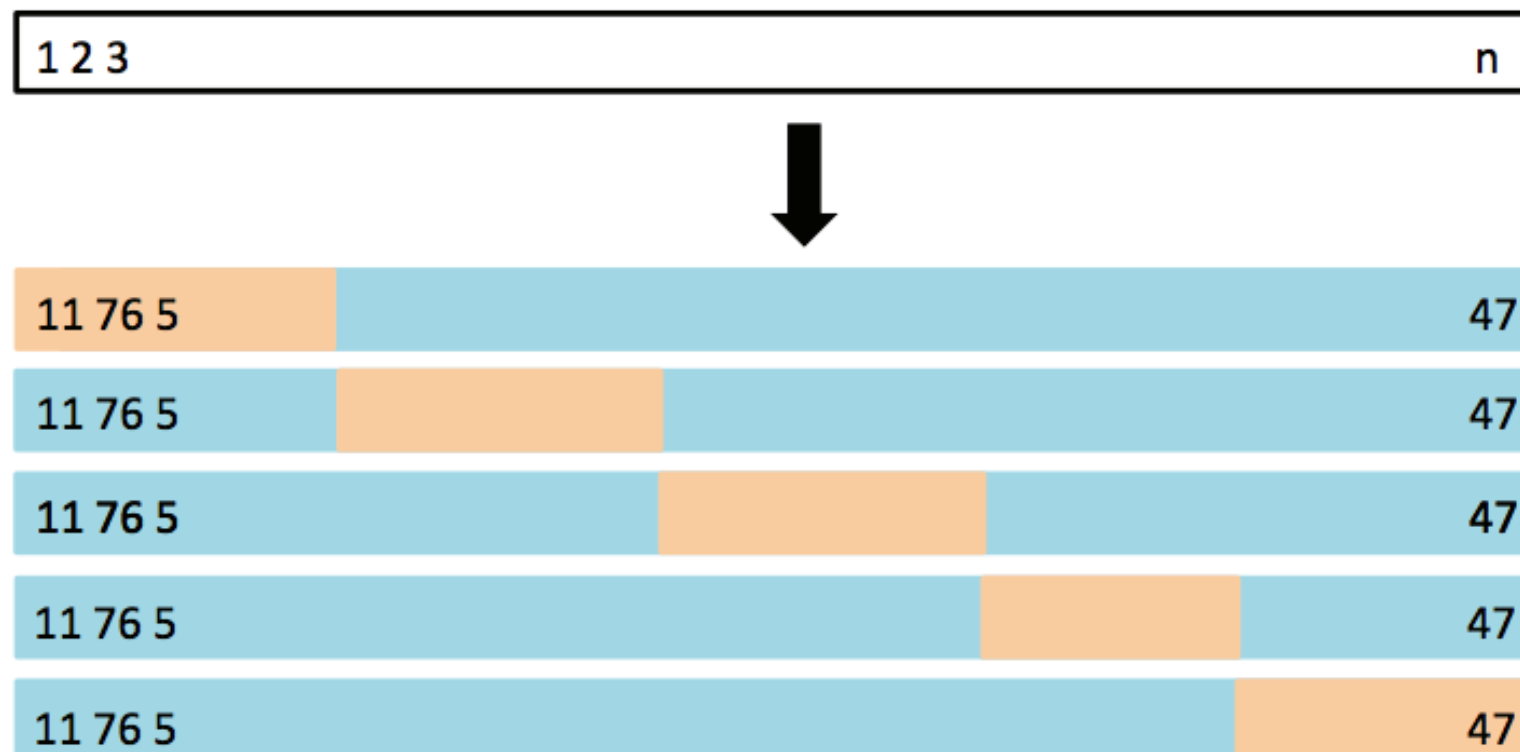
K-Fold Cross Validation

Split the data into k subsets or *folds*.

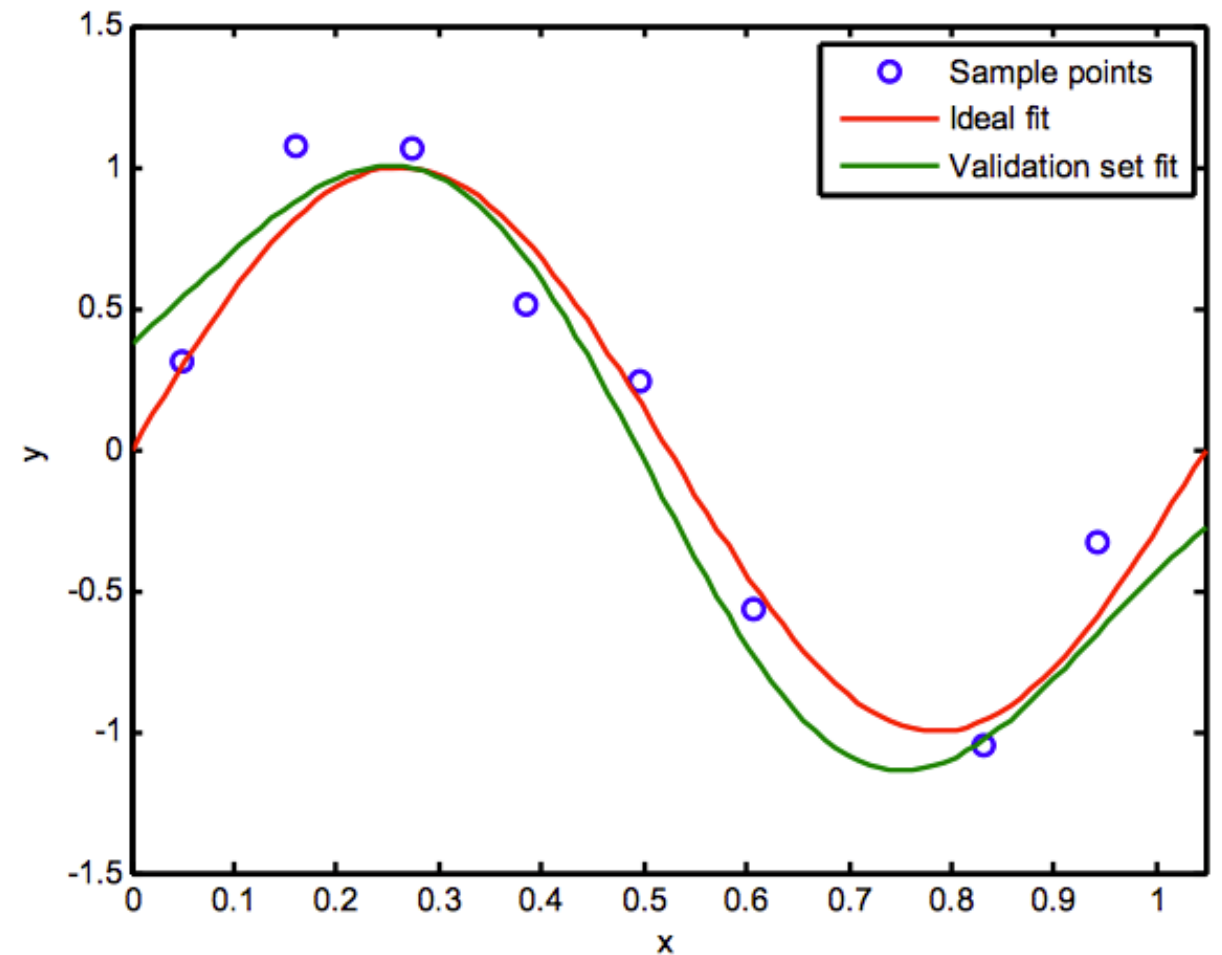
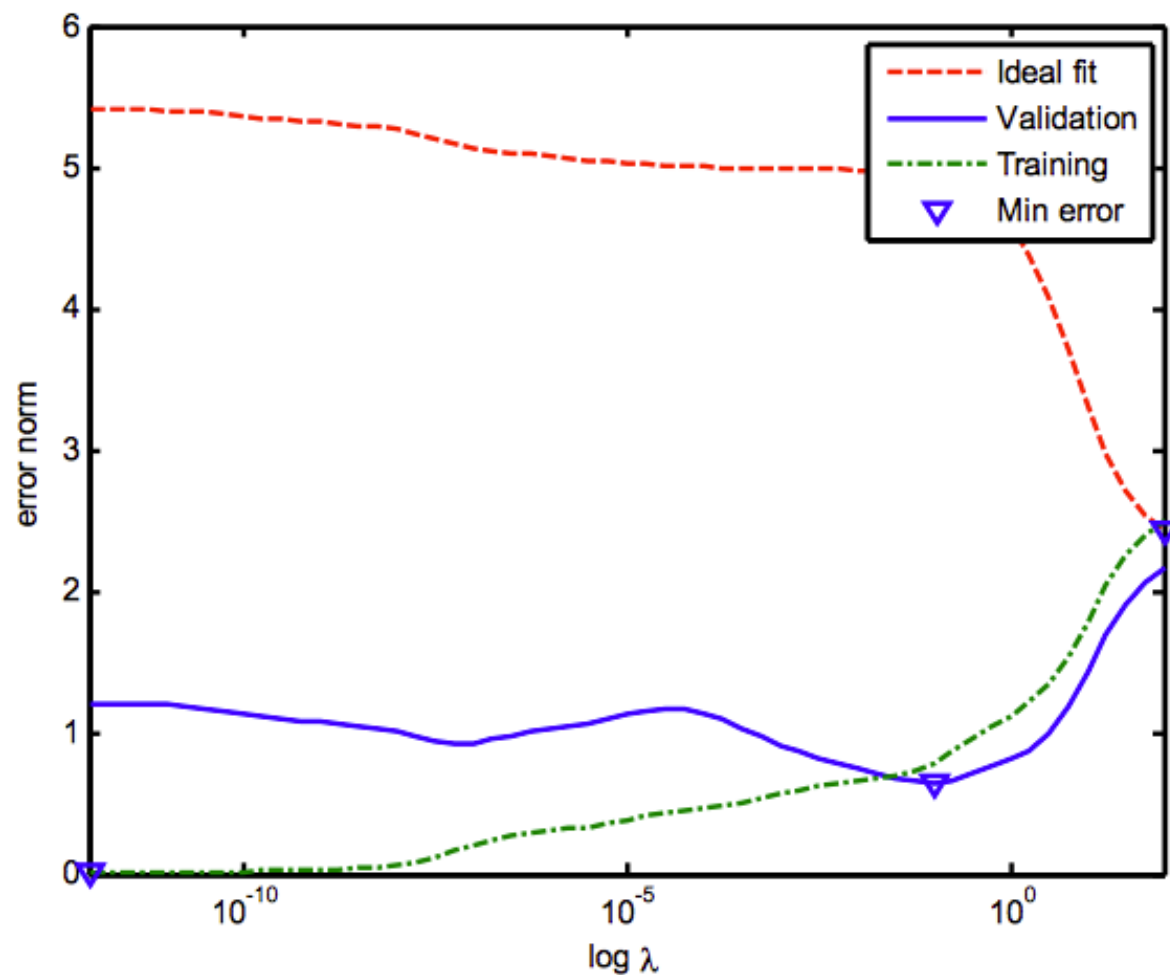
For every $i = 1, \dots, k$:

- ▶ train the model on every fold except the i th fold,
- ▶ compute the test error on the i th fold.

Average the test errors.



Choosing λ Using Validation Dataset



Pick up the lambda with the lowest mean value of rmse calculated by Cross Validation approach

Take-Home Messages

- What is overfitting
- What is regularization
- How does Ridge regression work
- Sparsity properties of Lasso regression
- How to choose the regularization coefficient λ