Georgia Tech

Naïve Bayes and Logistic Regression

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These slides are inspired based on slides from Le Song, Eric Eaton, and Chao Zhang.

The Moment

Incoming ML HW3

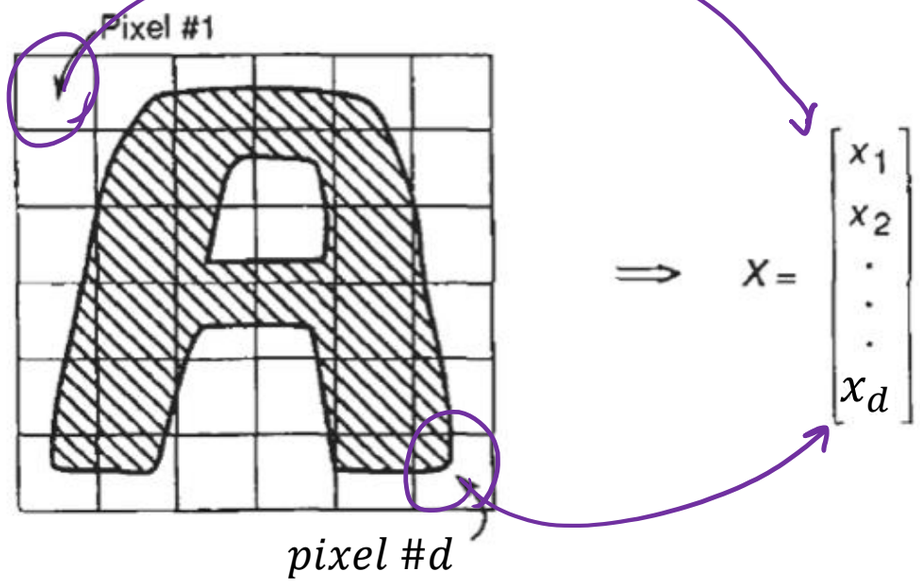
You; after HW2

Outline

- Generative and Discriminative Classification
- The Logistic Regression Model
- Understanding the Objective Function
- Gradient Descent for Parameter Learning
- Multiclass Logistic Regression

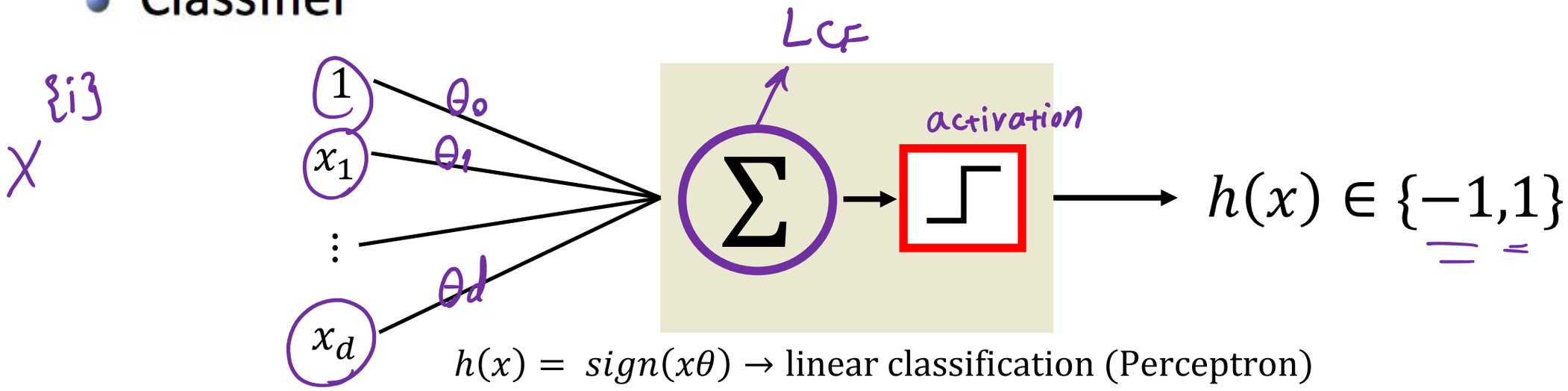
Classification

- Represent the data



- A label is provided for each data point, eg., $y \in \{-1, +1\}$

- Classifier

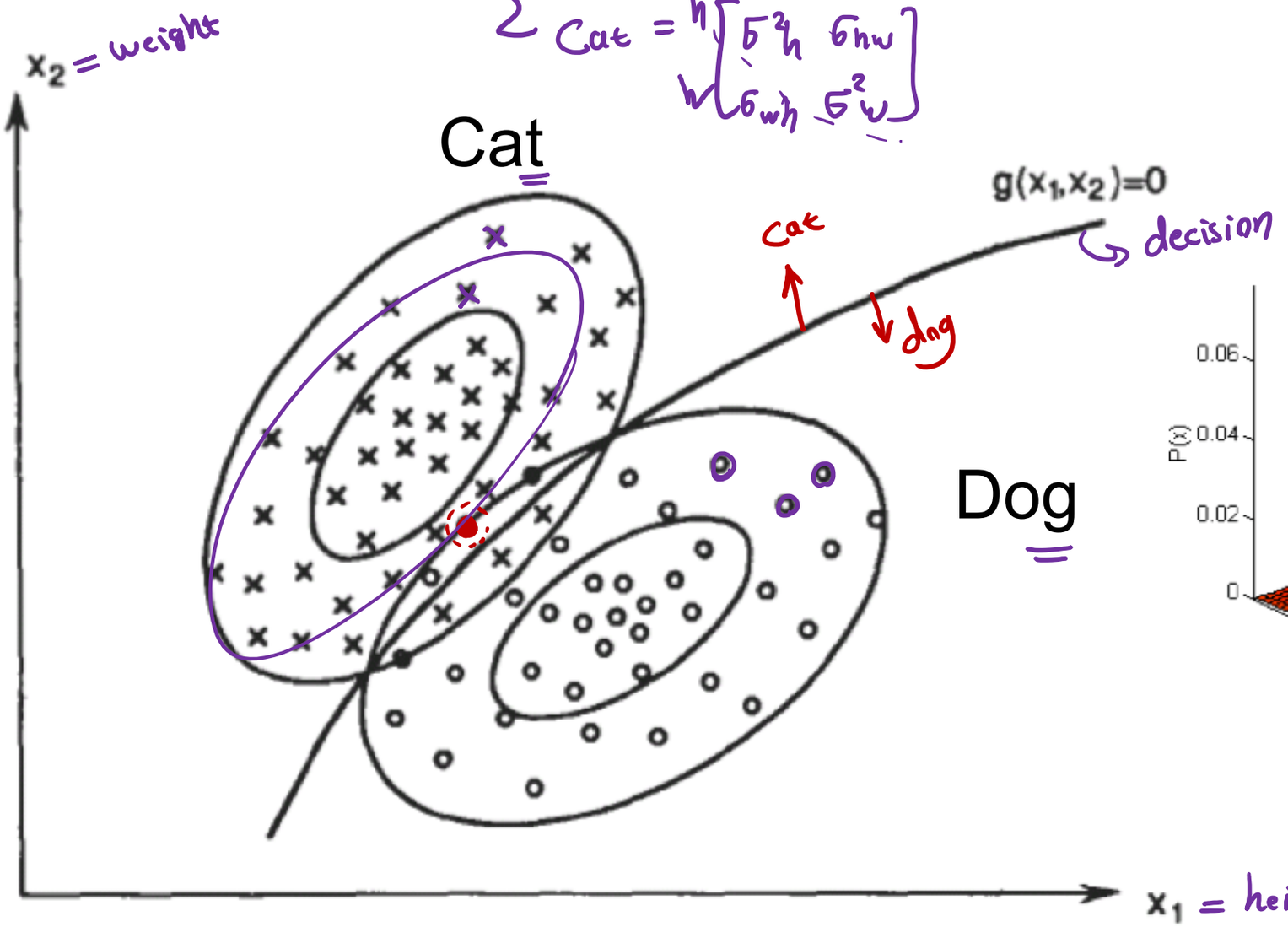


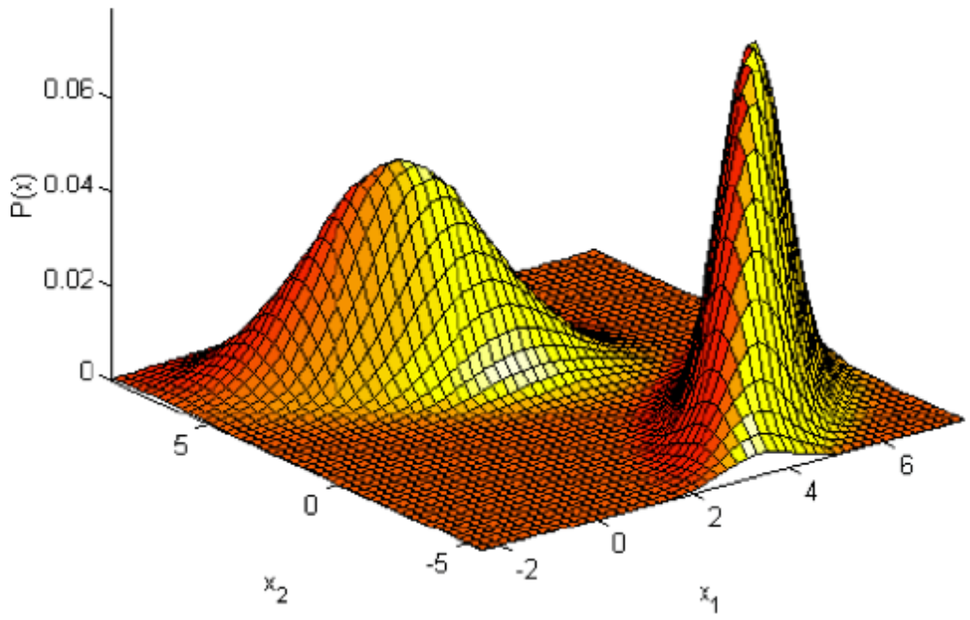
$h(x) = sign(x\theta) \rightarrow$ linear classification (Perceptron)

Decision Making: Dividing the Feature Space

- Distributions of sample from normal (positive class) and abnormal (negative class) tissues

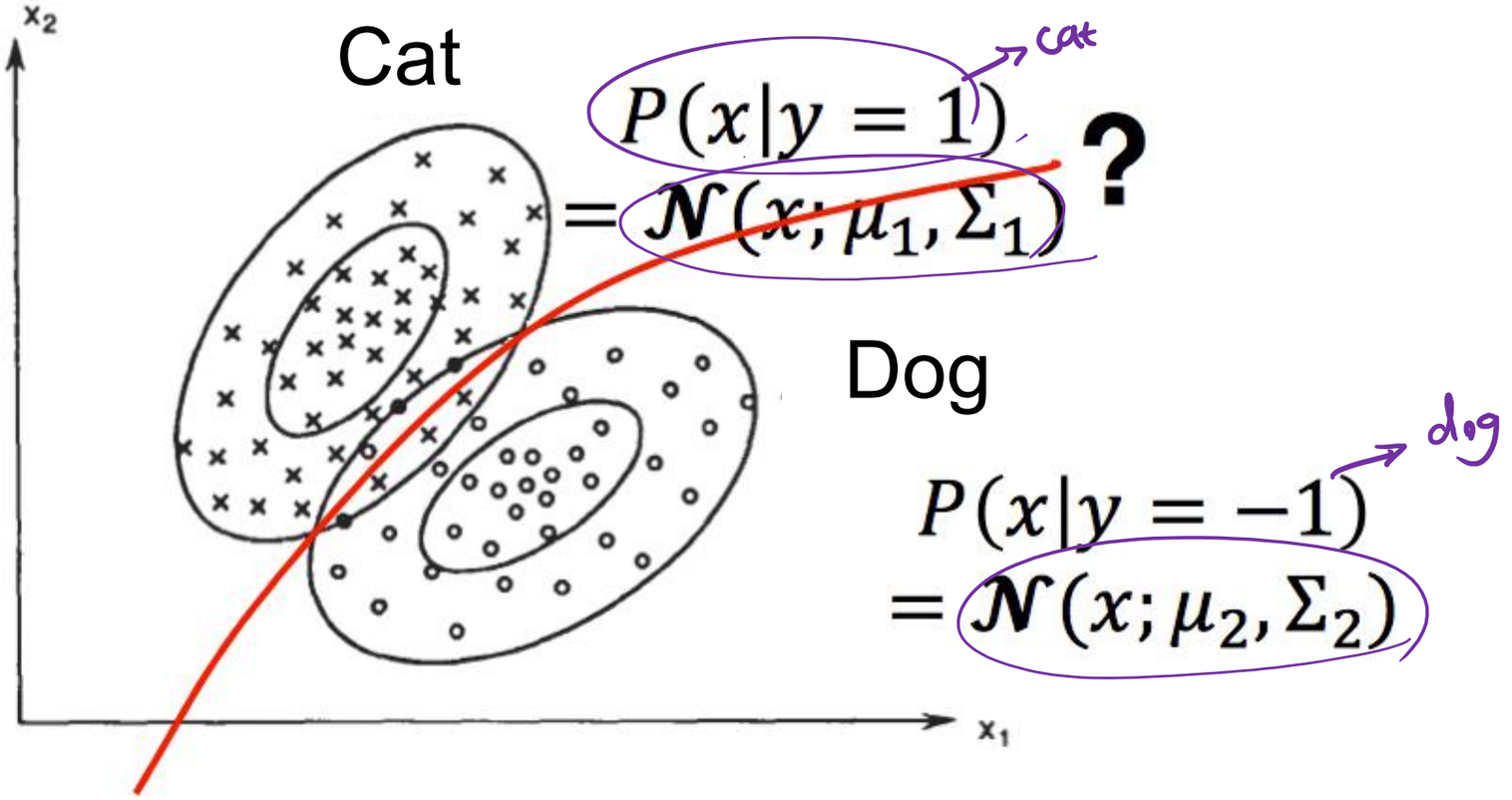
$$\Sigma_{Cat} = \begin{matrix} h \\ w \end{matrix} \begin{bmatrix} \sigma^2_h & \sigma_{hw} \\ \sigma_{wh} & \sigma^2_w \end{bmatrix} \begin{matrix} h & w \end{matrix}$$





How to Determine the Decision Boundary?

- Given class conditional distribution: $P(x|y=1), P(x|y=-1)$, and class prior: $P(y=1), P(y=-1)$



Bayes Decision Rule

$P(a,b,c) = P(a|b,c)\,P(b,c) = P(a|c)\,P(b|c)\,P(c)$

If a & b independent

likelihood Prior

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x,y)}{\sum_z P(x,y)}$$

posterior normalization constant

Prior: $P(y)$

Likelihood (class conditional distribution : $p(x|y) = \mathcal{N}(x|\mu_y, \Sigma_y)$

Posterior: $P(y|x) = \frac{P(y)\mathcal{N}(x|\mu_y, \Sigma_y)}{\sum_y P(y)\mathcal{N}(x|\mu_y, \Sigma_y)}$

Bayes Decision Rule

- Learning: prior: $p(y)$,class conditional distribution : $p(x|y)$

- The poster probability of a test point

$$q_i(x) \coloneqq P(y = i|x) = \frac{P(x|y)P(y)}{P(x)}$$

- Bayes decision rule:
  - If $q_i(x) > q_j(x)$, then $y = i$, otherwise $y = j$

- Alternatively:
  - If ratio $l(x) = \frac{P(x|y=i)}{P(x|y=j)} > \frac{P(y=j)}{P(y=i)}$, then $y = i$, otherwise $y = j$
  - Or look at the log-likelihood ratio $\mathrm{h(x)} = -\ln \frac{q_i(x)}{q_j(x)}$

$$x^{\{i\}}$$

$$\underbrace{P(y=+1 \mid x^{\{i\}})}_{cat} + \underbrace{P(y=-1 \mid x^{\{i\}})}_{dog} = 1$$

$$P(y=+1 \mid x^{\{i\}}) = \frac{\overbrace{P(x^{\{i\}} \mid y=+1)}^{\text{Likelihood}} \overbrace{P(y=+1)}^{\text{prior}}}{P(x^{\{i\}})}$$

$$X = \begin{matrix} x^{\{1\}} \\ x^{\{2\}} \\ x^{\{3\}} \\ x^{\{4\}} \\ x^{\{5\}} \\ x^{\{6\}} \end{matrix} \begin{bmatrix} h & w \\ \text{cat} & \\ \text{cat} & \\ \text{cat} & \\ \text{dog} & \\ \text{dog} & \\ \text{dog} & \end{bmatrix} \qquad y = \begin{bmatrix} \text{cat} \\ \text{cat} \\ \text{cat} \\ \text{dog} \\ \text{dog} \\ \text{dog} \end{bmatrix}$$

$$\frac{3}{6}$$

$$\Sigma_{cat} = \frac{1}{N} \bar{X}_{cat}^{T} \bar{X}_{cat}$$

$$\mu_{cat} = [\mu_{h\,cat}, \mu_{w\,cat}]$$

$$P(x^{\{i\}} \mid y=+1) = \frac{1}{\sqrt{2\pi |\Sigma|^2}_{cat}} \exp\left(- \underbrace{(x^{\{i\}} - \mu)}_{(1\times d)} \underbrace{\Sigma^{-1}}_{d\times d} \underbrace{(x^{\{i\}} - \mu)^T}_{d\times 1} \right)$$

$$\underbrace{P(x^{\{i\}} \mid y=+1)}_{1\times 1}$$

$$\frac{a}{a+b}, \frac{b}{a+b}$$

$$P(x^{\{i\}}) = \sum_{y=\pm 1} P(x^{\{i\}}, y=y) = \underbrace{P(x^{\{i\}}, y=+1)}_{cat} + \underbrace{P(x^{\{i\}}, y=-1)}_{dog} = P(x^{\{i\}} \mid y=+1) P(y=+1) + P(x^{\{i\}} \mid y=-1) P(y=-1)$$

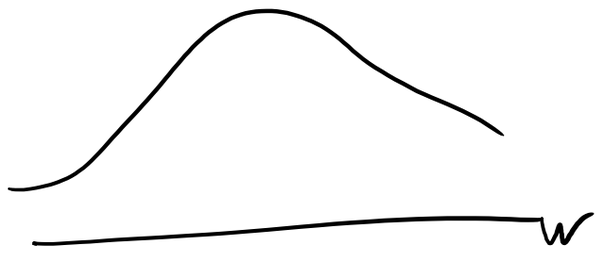
Naïve ⇝ the features are assumed to be conditionally independent $a \perp b$

$$p(a,b|c) = p(a|c)\,p(b|c)$$

$$p(\underline{x}^{\{i\}} \mid y=+1) \Rightarrow p(\underline{x}_1^{\{i\}}, \underline{x}_2^{\{i\}} \mid y=+1) = p(x_1^{\{i\}} \mid y=+1)\, p(x_2^{\{i\}} \mid y=+1)$$

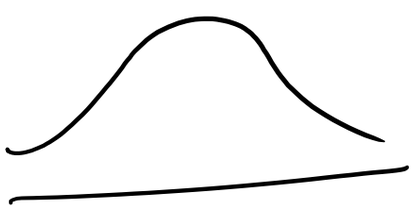
$$\Sigma = \begin{matrix} & h & w \\ h & \begin{bmatrix} \sigma_h^2 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} \end{matrix} \qquad \Sigma^{-1} = \frac{1}{\Sigma}$$

$p(a,b,c)$ where $a \perp b$



$$p(y=+1 \mid x^{\{i\}}) = \frac{p(x^{\{i\}}, y=+1)}{p(x^{\{i\}})} = \frac{p(x_1^{\{i\}}, x_2^{\{i\}}, y=+1)}{p(x^{\{i\}})} =$$

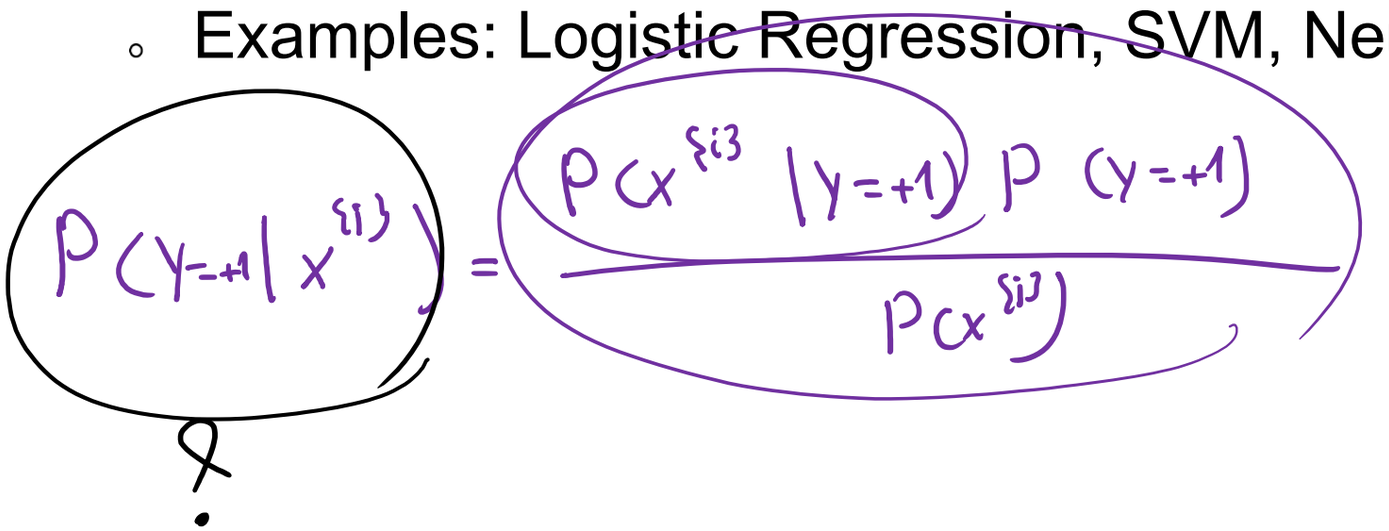
$$= \frac{p(x_1^{\{i\}} \mid y=+1)\, p(x_2^{\{i\}} \mid y=+1)\, p(y=+1)}{p(x^{\{i\}})}$$



Generative model

What do People do in Practice?

- Generative models
  - Model prior and likelihood explicitly
  - “Generative” means able to generate synthetic data points
  - Examples: Naive Bayes, Hidden Markov Models
- Discriminative models
  - Directly estimate the posterior probabilities
  - No need to model underlying prior and likelihood distributions
  - Examples: Logistic Regression, SVM, Neural Networks



Generative Model: Naive Bayes

- Use Bayes decision rule for classification

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

- But assume $p(x|y=1)$ is fully factorized : Dimensions are independent.

Conditionally

$$p(x|y=1) = \prod_{i=1}^{d} p(x_i|y=1)$$

- Or the variables corresponding to each dimension of the data are independent given the label

“Naïve” conditional independence assumption

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x,y)}{P(x)}$$

Joint probability model:

$$P(x, y_{label=1}) = \mathrm{P}(\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}}) = \mathrm{P}(\mathrm{x}_1|\mathrm{x}_2, , \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})P(\mathrm{x}_2, \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})$$

$$= \mathrm{P}(\mathrm{x}_1|\mathrm{x}_2, , \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})P(\mathrm{x}_2|\mathrm{x}_3 \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})P(\mathrm{x}_3, \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})$$

$$= \cdots$$

$$= \mathrm{P}(\mathrm{x}_1|\mathrm{x}_2, , \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}})P(\mathrm{x}_2|\mathrm{x}_3 \ldots, \mathrm{x}_\mathrm{d}, \mathrm{y}_{\mathrm{label=1}}) \ldots P(\mathrm{x}_{\mathrm{d}-1}|x_d, y_{label=1})P(x_d|y_{label=1})\mathrm{P}(y_{label=1})$$

Naïve Bayes assumption: let’s rewrite it as:

$$P(x, y_{label=1}) = P(x_1|y_{label=1})P(x_2|y_{label=1}) \ldots P(x_d|y_{label=1})P(y_{label=1}) =$$

$$P(y_{label=1}) \prod_{i=1}^{d} P(x_i|y_{label=1})$$

Gaussian naïve Bayes
A typical assumption

Example

$x^{[i]}_{1 \times d+1}$ $\quad y=+1 \rightarrow$ cat $\quad y=-1 \rightarrow$ dog

Multivariate

$X^{[i]} = [x^{[i]}_1, x^{[i]}_2]$

height weight

$$P(y=+1 \mid x^{[i]}) = \frac{\overbrace{P(x^{[i]} \mid y=+1)}^{\text{Likelihood}} \overbrace{P(y=+1)}^{\text{Prior}}}{P(x^{[i]})} \rightarrow N(x \mid \mu, \Sigma)$$

$P(x^{[i]} \mid y=+1) = P(x^{[i]}_1, x^{[i]}_2 \mid y=+1) \Rightarrow$ conditionally independce $\Rightarrow P(x_1^{[i]} \mid y=+1)\, P(x_2^{[i]} \mid y=+1)$

Univorciate

$N(x \mid \mu, \sigma)$

$P(x^{[i]} \mid y=-1)$

$P(y=+1 \mid x^{[i]}) = ?$

Discriminative Models

- Directly estimate decision boundary $\mathrm{h}(\mathrm{x}) = -\ln \frac{q_i(x)}{q_j(x)}$ or posterior distribution $p(y|x)$
  - Logistic regression, Neural networks
  - Do not estimate $p(x|y)$ and $p(y)$

- Why discriminative classifier?
  - Avoid difficult density estimation problem → Generative model
  - Empirically achieve better classification results

Outline

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- Gradient Descent for Parameter Learning
- Multiclass Logistic Regression

Gaussian Naïve Bayes

$$P(y=1|x) = \frac{P(x|y=1)P(y=1)}{P(x)} = \frac{P(y=1)\prod_{i=1}^{d} P(x_i|y=1)}{P(x)}$$

$$P(x) = \sum_{y=y_{cat\ \&\ dog}} P(x, y=y) = P(x, y=+1) + P(x, y=-1)$$

$$\prod_{i=1}^{d} p(x_i|y=1, \mu_{1i}, \sigma_{1i})$$

$$= \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi}\sigma_{1i}} \exp\left(-\frac{1}{2\sigma_{1i}^2}(x_{1i} - \mu_{1i})^2\right)$$

Prior: $p(y=1) = \pi_1$

Posterior: $p(y = 1 \mid x, \mu, \sigma, \pi)$

$$= \frac{\pi_1 \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi}\sigma_{1i}} \exp\left(-\frac{1}{2\sigma_{1i}^2}(x_i - \mu_{1i})^2\right)}{\sum_{k=1}^{2} \pi_k \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi}\sigma_{ki}} \exp\left(-\frac{1}{2\sigma_{ki}^2}(x_i - \mu_{ki})^2\right)}$$

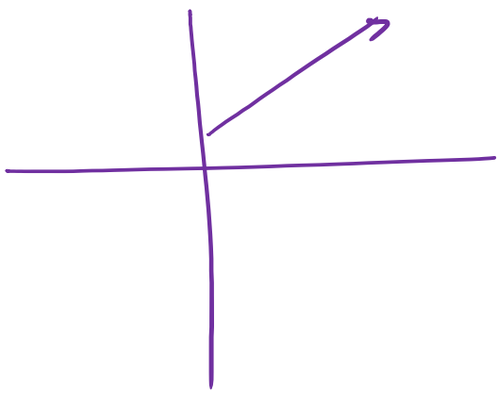
labels

get $\exp(\ln(u))$ of numerator and denominator

$$= \frac{\exp\left(-\sum_{i=1}^{d}\left(\frac{1}{2\sigma_{1i}^2}(x_i - \mu_{1i})^2 + \log\sigma_{1i} + C\right) + \log\pi_1\right)}{\sum_{k=1}^{2} \exp\left(-\sum_{i=1}^{d}\left(\frac{1}{2\sigma_{ki}^2}(x_i - \mu_{ki})^2 + \log\sigma_{ki} + C\right) + \log\pi_k\right)}$$

$$\frac{a}{a+b} \rightsquigarrow \frac{\frac{a}{a}}{\frac{a}{a}+\frac{b}{a}} = \frac{1}{1+\frac{b}{a}}$$

$$= \frac{\exp\left(-\sum_{i=1}^{d}\left(\frac{1}{2\sigma_i^2}(x_i-\mu_{1i})^2+\log\sigma_i+C\right)+\log\pi_1\right)}{\sum_{k=1}^{2}\exp\left(-\sum_{i=1}^{d}\left(\frac{1}{2\sigma_i^2}(x_i-\mu_{ki})^2+\log\sigma_i+C\right)+\log\pi_k\right)}$$



$y = \theta_0 + \theta_1 x_1$

$$= \frac{1}{1+\exp\left(-\sum_{i=1}^{d}\left(x_i\frac{1}{\sigma_i}(\mu_{1i}-\mu_{2i})+\frac{1}{\sigma_i^2}\left(\mu_{1i}^2-\mu_{2i}^2\right)\right)+\log\frac{\pi_2}{\pi_1}\right)}$$

$\sum_i \theta_i x_i$ $\qquad$ θ_0

$$P(y=1|x) = \frac{1}{1+\exp\left(-\sum_{i=1}^{d}\left(x_i\frac{1}{\sigma_i}(\mu_{1i}-\mu_{2i})+\frac{1}{\sigma_i^2}(\mu_{1i}^2-\mu_{2i}^2)\right)+\log\frac{\pi_2}{\pi_1}\right)}$$

$\theta_1 = \frac{1}{\sigma_1}(\mu_{11}-\mu_{21})$

Number of parameters:

$2d+1 \rightarrow d\ mean, d\ variance, and\ 1\ for\ prior$

Sigmoid

$$P(y=1|x) = \frac{1}{1+\exp[-(\sum_{i=1}^{d}(\theta_i x_i)+\theta_0)]} = \frac{1}{1+\exp(-s)}$$

$\theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d$ LCF $= s = x\theta$

1

0

Number of parameters = $d+1 \rightarrow \theta_0, \theta_1, \theta_2, \ldots, \theta_d$

Why not directly learning $P(y=1|x)$ or θ parameters?

Gaussian Naïve Bayes is a subset of logistic regression

Why $\frac{1}{1+\exp(-x\theta)}$ is a probability?

Cat = 1/2 dog = 1/2

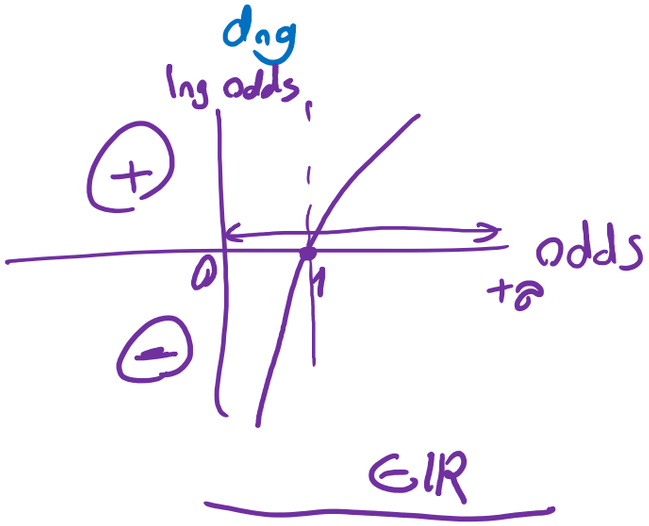
Cat = 3/4 dog = 1/4

Cat = 200/201, 1/201 dog

+∞

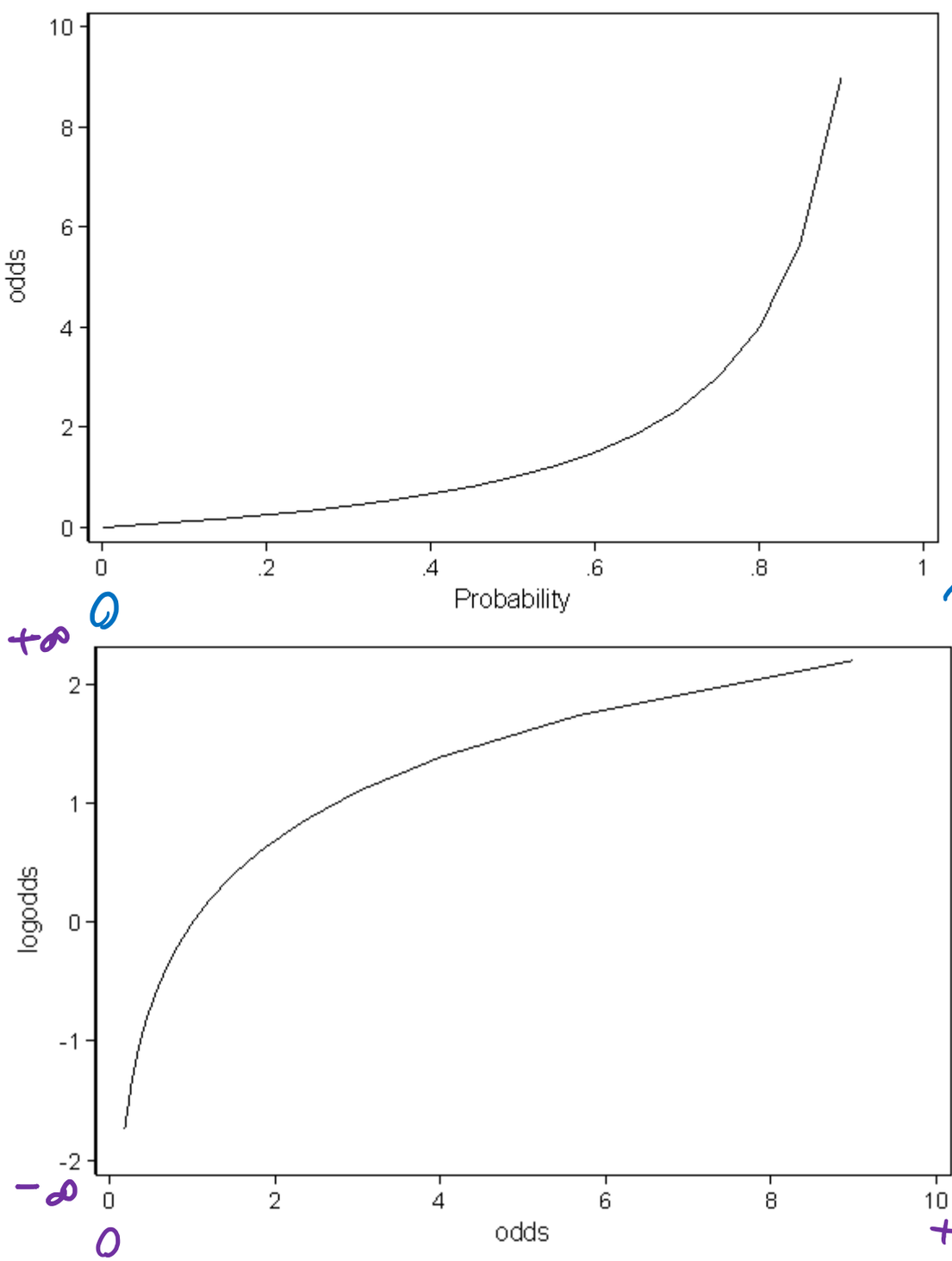
$\frac{P(y=1|x)}{1-P(y=1|x)}$ is called Odds

Cat



$\in \mathbb{R}$

$log(odds)\ vs\ odds$



What could be $x\theta$ domain?

$\theta_0 + \theta_1 x_1^{(i)} + \dots + \theta_d x_d^{(i)} \in \mathbb{R}$

What is logit function?

$$logit(p) = log(odds) = \log(\frac{p}{1-p})$$

$$\log\left(\frac{p}{1-p}\right) = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d = \sum_{i=0}^{d} x_i \theta_i = x\theta$$

$$exp\left(log\left(\frac{p}{1-p}\right)\right) = \exp(x\theta)$$

$$p = \frac{e^{x\theta}}{1 + e^{x\theta}} = \frac{1}{1 + e^{-x\theta}}$$

Logistic function for posterior probability

θ ✓

$$P(y=1|x^{[i]}) = \frac{1}{1+\exp(-x^{[i]}\theta)} \rightsquigarrow 0.6 \quad \text{Cat}$$

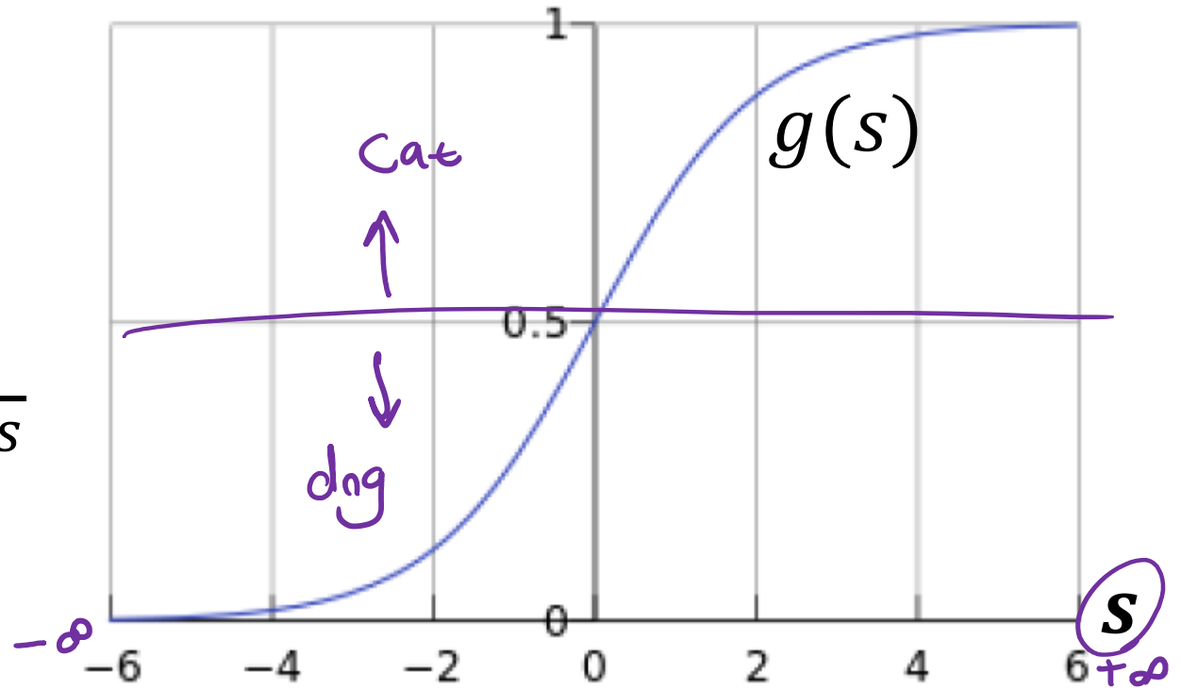
$$P(y=1|x) = \frac{1}{1+\exp(-x\theta)}$$

Many equations can give us this shape

Let's use the following function:

$$s = x\theta$$

$$g(s) = P(y=1|x) = \frac{e^s}{1+e^s} = \frac{1}{1+e^{-s}}$$



This formula is called sigmoid function

It is easier to use this function for optimization

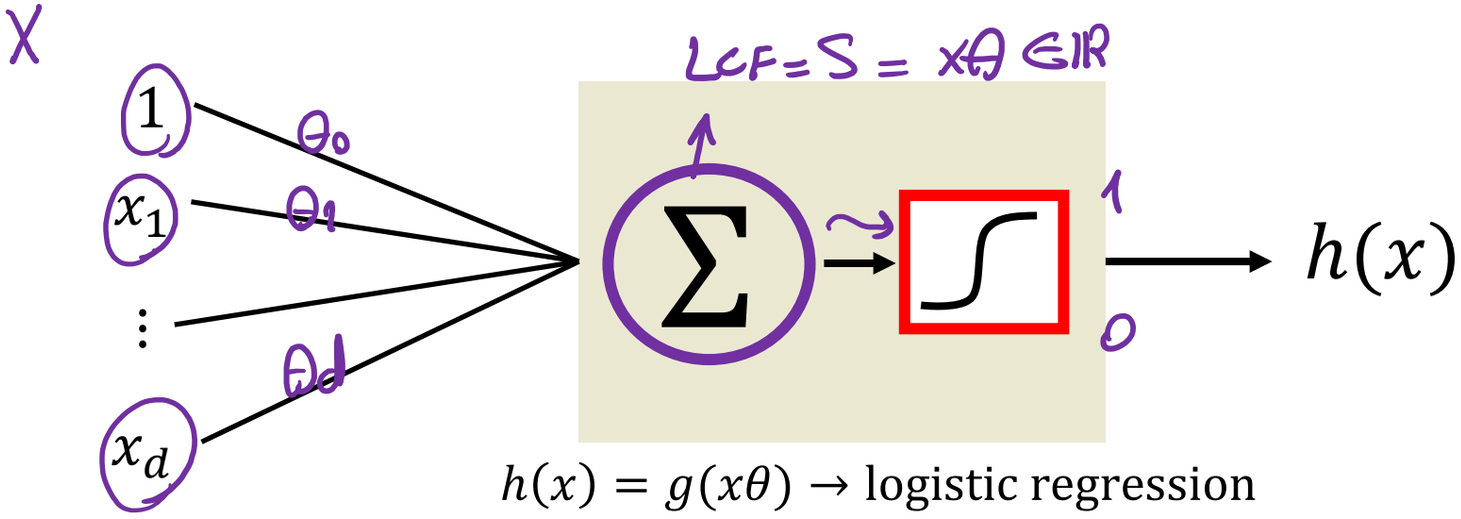
Is 0.5 threshold cut-off a good choice?

Learn about ROC and AUC (False positive rate and True positive rate) (Interactive)

Sigmoid Function

$$g(s) = \frac{e^s}{1+e^s} = \frac{1}{1+e^{-s}}$$

$$s = \sum_{i=0}^{d} x_i \theta_i = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d$$

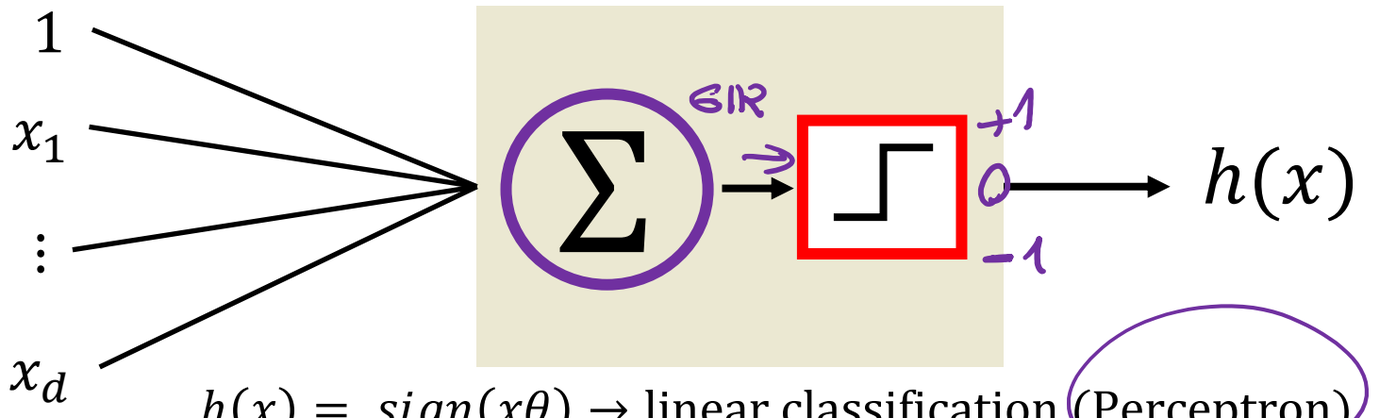


$h(x) = g(x\theta) \rightarrow$ logistic regression

Soft classification
Posterior probability

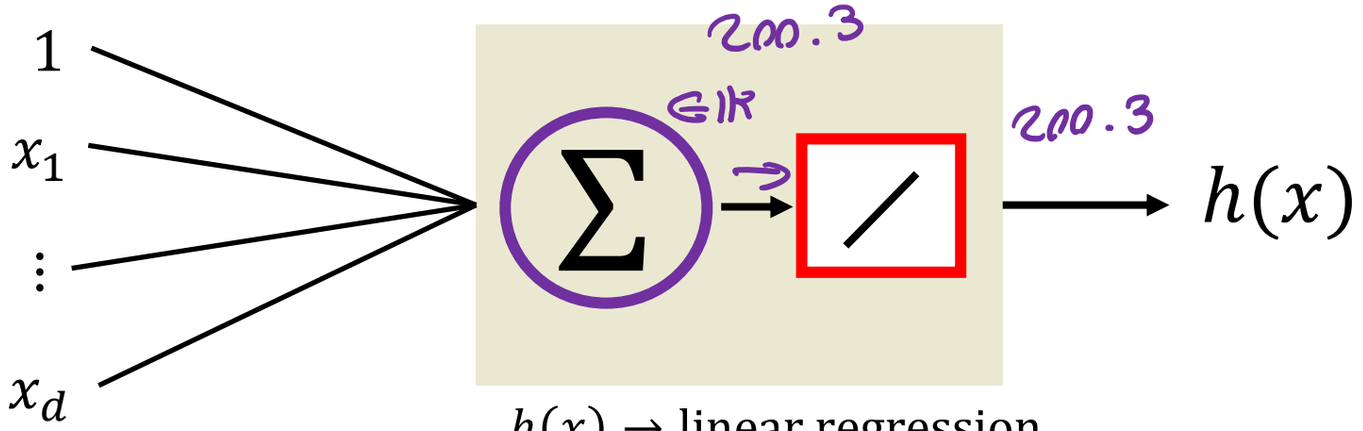
$$s = \sum_{i=0}^{d} x_i \theta_i = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d$$

Three linear models

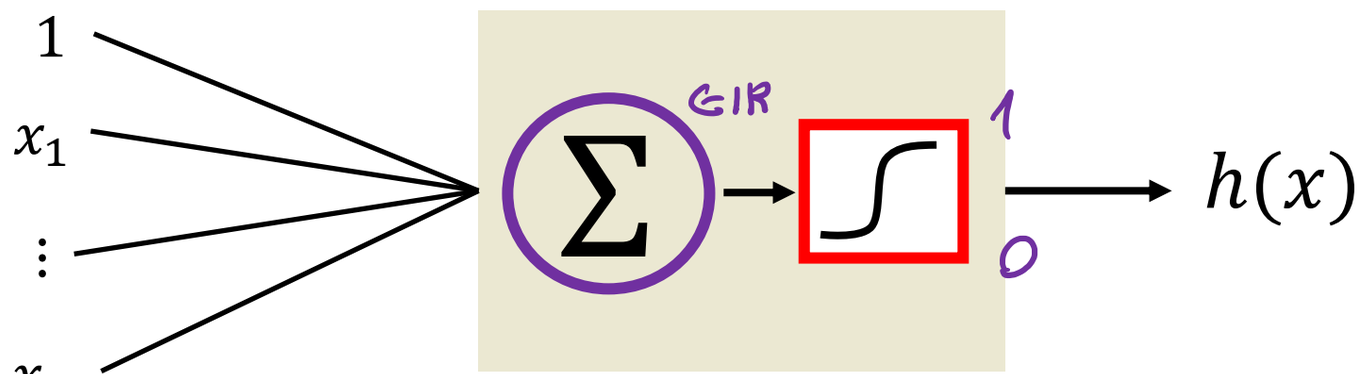


Hard classification

$h(x) = sign(x\theta) \rightarrow$ linear classification (Perceptron)



$h(x) \rightarrow$ linear regression



Soft classification
Posterior probability

$h(x) = g(x\theta) \rightarrow$ logistic regression

$g(s)$ is interpreted as probability

Example: Prediction of heart attacks

Input $\boldsymbol{x}$: cholesterol level, age, weight, finger size, etc.

$g(s)$: probability of heart attack within a certain time

We can't have a hard prediction here

$s = x\theta$ Let's call this risk score

$$h_\theta(x) = p(y|x) = \begin{cases} g(s), & y = 1 \\ 1 - g(s), & y = 0 \end{cases}$$

$\hat{y}_p$ → $g(s)$, $1 - \hat{y}_p$ → $1 - g(s)$

$y = 1$ → cat

$y = 0$ → dog

Using posterior probability directly

Logistic regression model

$$p(y|x) = \begin{cases} \dfrac{1}{1+\exp(-x\theta)} & y = 1 \\ 1 - \dfrac{1}{1+\exp(-x\theta)} = \dfrac{\exp(-x\theta)}{1+\exp(-x\theta)} & y = 0 \end{cases}$$

We need to find θ parameters, let's set up log-likelihood for **n** datapoints

$$l(\theta) := log \prod_{i=1}^{n} p(y^{\{i\}}|x^{\{i\}}, \theta)$$

$$= \sum_{i} \theta^T (x^{\{i\}})^T \left(y^{\{i\}} - 1\right) - \log(1 + \exp(-x^{\{i\}}\theta))$$

This form is concave, negative of this form is convex

The gradient of $l(\theta)$

$$\theta^{t+1} \leftarrow \theta^{t} - \alpha \frac{\partial l(\theta)}{\partial \theta}$$

avg Negative log likelihood = CE

$$l(\theta) = -log \prod_{i=1}^{n} p(y^{\{i\}} | x^{\{i\}}, \theta)$$

$$= \sum_{i} \theta^T (x^{\{i\}})^T \left(y^{\{i\}} - 1\right) - \log(1 + \exp(-x^{\{i\}}\theta))$$

$$CE = -\sum_{i=1}^{N} y_a^{\{i\}} \log \hat{y}_p^{\{i\}} \qquad \nabla(CE) = X^T(\hat{y}_p - y_a)$$

- Gradient

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_{i} (x^{\{i\}})^T \left(y^{\{i\}} - 1\right) + (x^{\{i\}})^T \frac{\exp(-x^{\{i\}}\theta)}{1 + \exp(-x^{\{i\}}\theta)}$$

$$= -\sum (x^{\{i\}})^T (y_a^{\{i\}} - 1) + (x^{\{i\}})^T (1 - \hat{y}_p^{\{i\}})$$

$$= -\sum (x^{\{i\}})^T (y_a^{\{i\}} - \hat{y}_p^{\{i\}}) = \sum_{i=1}^{N} (x^{\{i\}})^T (\hat{y}_p^{\{i\}} - y_a^{\{i\}})$$

- Setting it to 0 does not lead to closed form solution

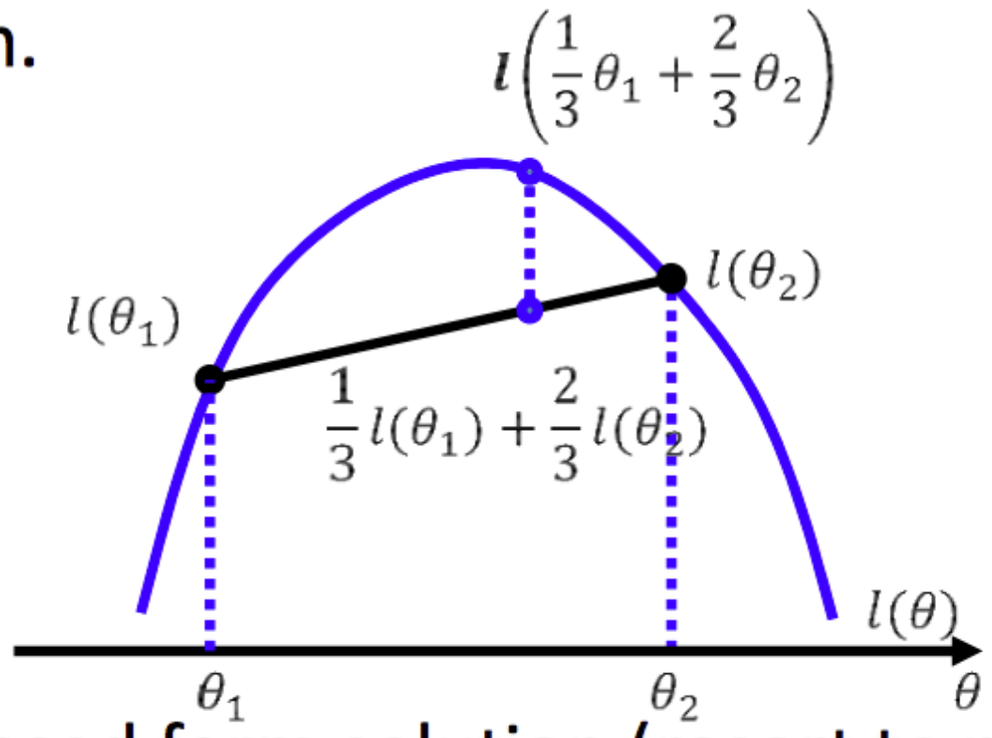
$$= X^T (y_p - y_a)$$

The Objective Function

- Find θ, such that the conditional likelihood of the labels is maximized

$$\max_{\theta} l(\theta) := log \prod_{i=1}^{n} p(y^{\{i\}}|x^{\{i\}}, \theta)$$

- Good news: $l(\theta)$ is concave function of θ, and there is a single global optimum.



- Bad new: no closed form solution (resort to numerical method)

Gradient Descent

- One way to solve an *unconstrained* optimization problem is gradient descent

$$\theta^{t+1} \leftarrow \theta^{t} - \alpha X^{T}(\hat{Y}_p - Y_a)$$

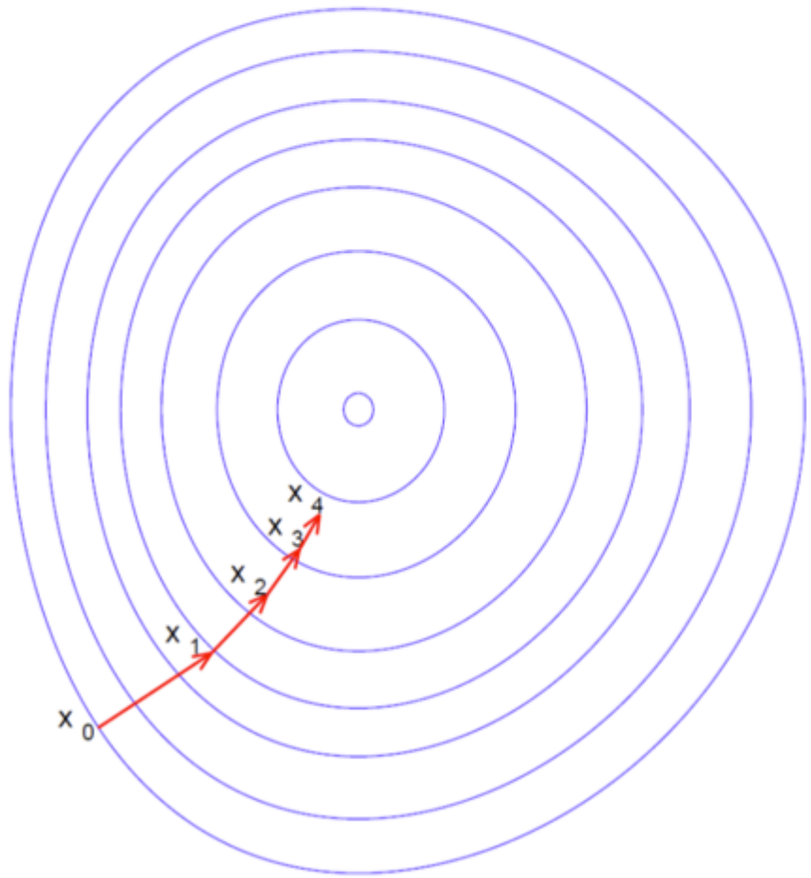
- Given an initial guess, we *iteratively* refine the guess by taking the direction of the negative gradient

- Think about going down a hill by taking the steepest direction at each step

- Update rule

$$x_{k+1} = x_k - \gamma_k \nabla f(x_k)$$

γ_k is called the step size or learning rate



Gradient Ascent(concave)/Descent(convex) algorithm

- Initialize parameter θ^0
- Do

$$\theta^{t+1} \leftarrow \theta^t + \eta \sum_i (x^{\{i\}})^T \left(y^{\{i\}} - 1\right) + (x^{\{i\}})^T \frac{\exp\left(-x^{\{i\}}\theta\right)}{1 + \exp(-x^{\{i\}}\theta)}$$

- While the $||\theta^{t+1} - \theta^t|| > \epsilon$

$\hat{y}_p = P(y=1|x) = \frac{1}{1+\exp(-x\theta)}$

$1 - \hat{y}_p = 1 - P(y=1|x)$

$\theta^{t+1} \leftarrow \theta^t - \alpha X^T (\hat{y}_p - \hat{y}_a)$

Softmax

$LCF \leftarrow S_1 = X\Theta_{cat} \qquad S_2 = X\Theta_{dog}$

$$P(y=+1 \mid x) = \frac{\exp(-S_1)}{\exp(-S_1)+\exp(-S_2)} \quad \text{cat}$$

$$P(y=-1 \mid x) = \frac{\exp(-S_2)}{\exp(-S_1)+\exp(-S_2)} \quad \text{dog}$$

$$\Theta = \begin{bmatrix} \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} \quad \begin{matrix} \Theta_{cat} & \Theta_{dog} \\ & \\ & \\ & \\ & \end{matrix} \qquad d+1 \times 2$$

$X^{[i]} \rightarrow \begin{bmatrix} \hat{y}_p & 1-\hat{y}_p \end{bmatrix}$

$$\underset{d+1\times 2}{\Theta^{t+1}} \leftarrow \underset{d+1\times 2}{\Theta^{t}} - \alpha \underset{d+1\times n}{X^T} \left(\overset{n\times 2}{\hat{Y}_p} - \overset{n\times 2}{Y_a} \right)$$

$[0.6 \; 0.4] \rightsquigarrow [1 \quad 0]$

$$P(y=+1 \mid x) = \frac{\frac{\exp(-s_1)}{\exp(-s_1)}}{\frac{\exp(-s_1)+\exp(-s_2)}{\exp(-s_1)}} = \frac{1}{1+\exp(s_1-s_2)} \qquad \text{Ref} = \frac{1}{1+\exp(-x\theta)}$$

$$P(y=-1 \mid x) = \frac{\frac{\exp(-s_2)}{\exp(-s_1)}}{1+\exp(s_1-s_2)} = \frac{\exp(s_1-s_2)}{1+\exp(s_1-s_2)} = \frac{\exp(-x\theta)}{1+\exp(-x\theta)}$$

$s_1 = x\theta_{cat}$ $\quad s_2 = x\theta_{dog} \Rightarrow \quad s_1 - s_2 = x(\theta_{cat} - \theta_{dog}) = x\theta$

$s_2 - s_1 = -x\theta$

$$\theta^{t+1} \leftarrow \theta^{t} - \alpha X^{T}(y_p - y_a)$$

θ: $d+1 \times 1$ $\quad$ θ^t: $s_1 - s_2$

Multi class classification cat, dog, fish

$$P(Y=1 \mid X) = \frac{\exp(-s_1)}{\exp(-s_1)+\exp(-s_2)+\exp(-s_3)}$$

cat

dog

$$P(Y=2 \mid X) = \frac{\exp(-s_2)}{\dots}$$

fish

$$P(Y=3 \mid X) = \frac{\exp(-s_3)}{\dots}$$

$$s_3 = X\theta_{fish}$$

$$s_1 = X\theta_{cat} \qquad s_2 = X\theta_{dog}$$

$$\theta = \begin{bmatrix} \theta_{cat} & \theta_{dog} & \theta_{fish} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}_{d+1 \times 3}$$

$$\theta^{t+1}_{d+1\times 3} \leftarrow \theta^{t} - \alpha X^{T}(\hat{Y}_p - Y_a)$$

$\hat{Y}_p$: $n \times 3$; Y_a: $n \times 3$

$[0.6 \;\; 0.2 \;\; 0.2]$ → $[1 \;\; 0 \;\; 0]$

cat $[1 \;\; 0 \;\; 0]$

dog $[0 \;\; 1 \;\; 0]$

fish $[0 \;\; 0 \;\; 1]$

$$P(Y=1|X) = \frac{1}{1 + \exp(S_1 - S_2) + \exp(S_1 - S_3)}$$

Ref 1 Ref 2

$$P(Y=2|X) = \frac{\exp(S_1 - S_2)}{\text{(same)}}$$ ↙ dog

$$P(Y=3|X) = \frac{\exp(S_1 - S_3)}{\text{(same)}}$$ ↙ fish

$$S_1 - S_2 = X\theta_{cat} - X\theta_{dog} = X\theta_{Ref\ 1}$$

$$S_1 - S_3 = X\theta_{cat} - X\theta_{fish} = X\theta_{Ref\ 2}$$

$$\theta = \begin{bmatrix} \text{Ref 1} & \text{Ref 2} \\ \vdots & \vdots \\ & \end{bmatrix}$$

$$\underset{d+1 \times 2}{\theta^{t+1}} \leftarrow \theta^{t} - \alpha X^{T}(\hat{Y}_p - Y_a)$$

[0.6 0.4] [1 0]

Multi label classification

$X^{\{i\}}$ → movie $\quad Y_a^{\{i\}} = [\underbrace{1}_{\text{action}} \quad \underbrace{1}_{\text{adventure}} \quad \underbrace{0}_{\text{Sci-fi}}]$

$$\hat{Y}_p^{\{i\}} = \left[\underbrace{\frac{1}{1+\exp(-X\theta)}}_{\text{action}}, \underbrace{\frac{1}{1+\exp(-X\theta)}}_{\text{adventure}}, \underbrace{\frac{1}{1+\exp(-X\theta)}}_{\text{Sci-fi}}\right]$$

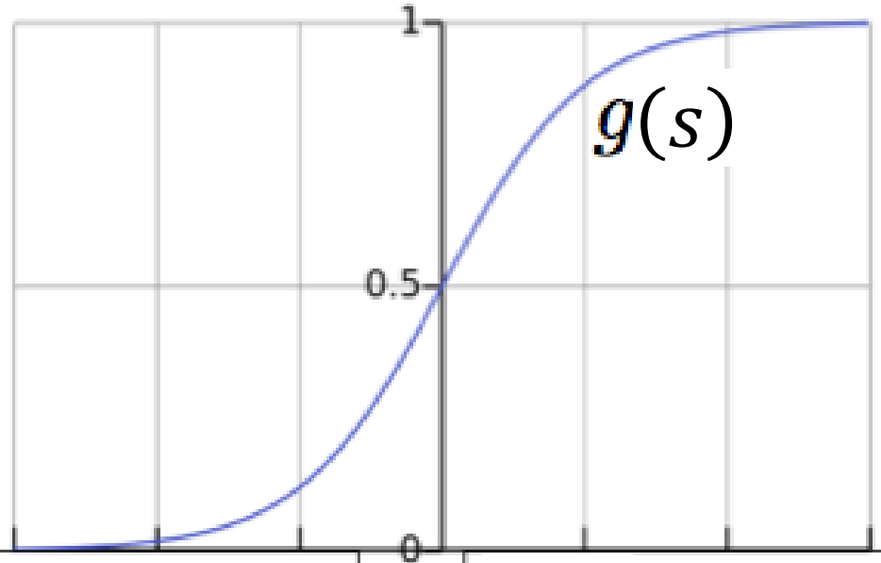
$$\theta^{t+1} \leftarrow \theta^{t} - \alpha X^{T}(\hat{Y}_p - Y_a)$$

$$\theta = \begin{bmatrix} \theta_{\text{action}} & \theta_{\text{adv}} & \theta_{\text{Sci-fi}} \\ & & \end{bmatrix}$$

Logistic Regression

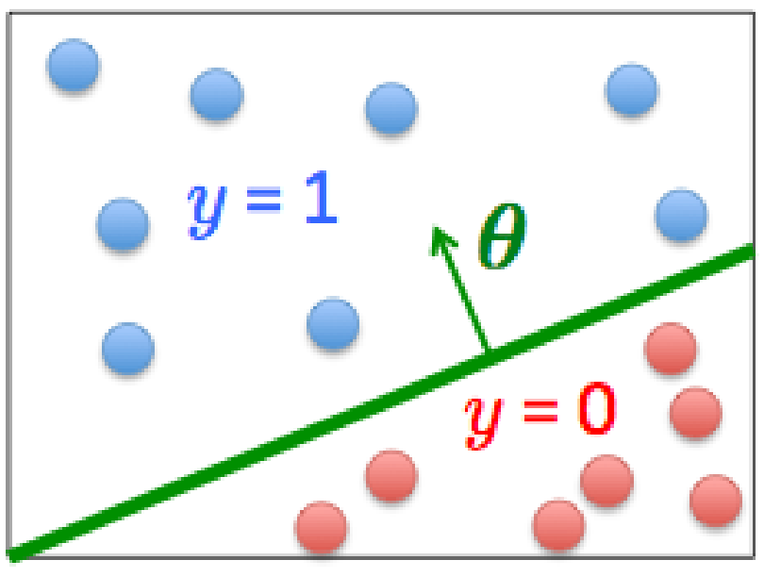
$$g(s) = \frac{e^s}{1+e^s} = \frac{1}{1+e^{-s}}$$

$$s = x\theta$$



$x\theta$ should be large negative values for negative instances	$x\theta$ should be large positive values for positive instances

- Assume a threshold and...
  - Predict $y = 1$ if $g(s) \geq 0.5$
  - Predict $y = 0$ if $g(s) < 0.5$

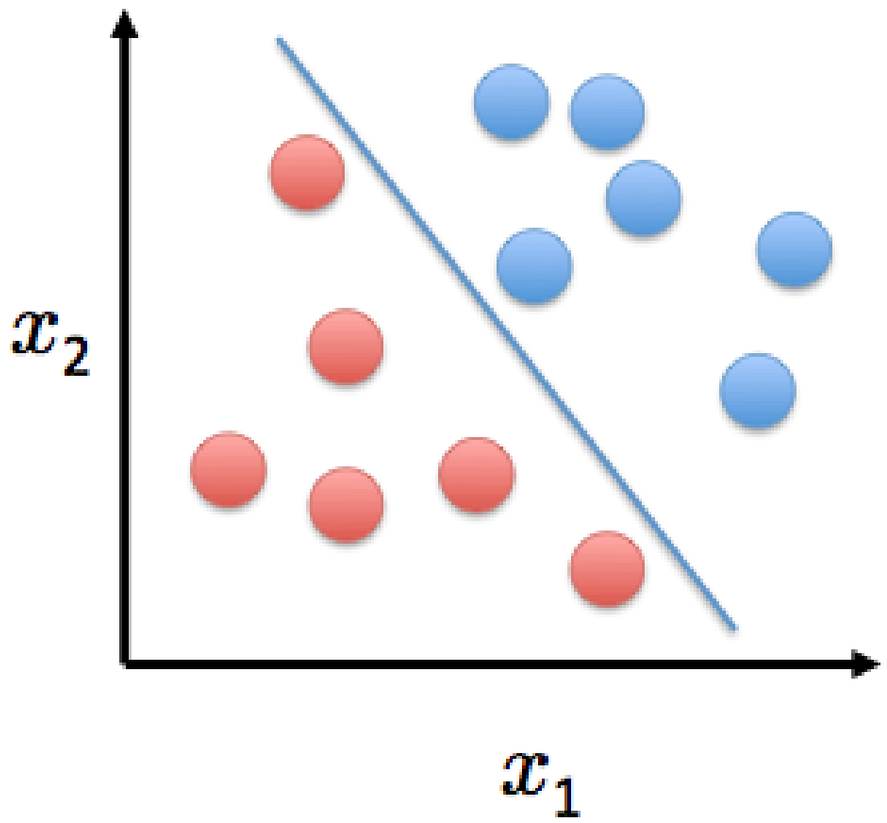


Outline

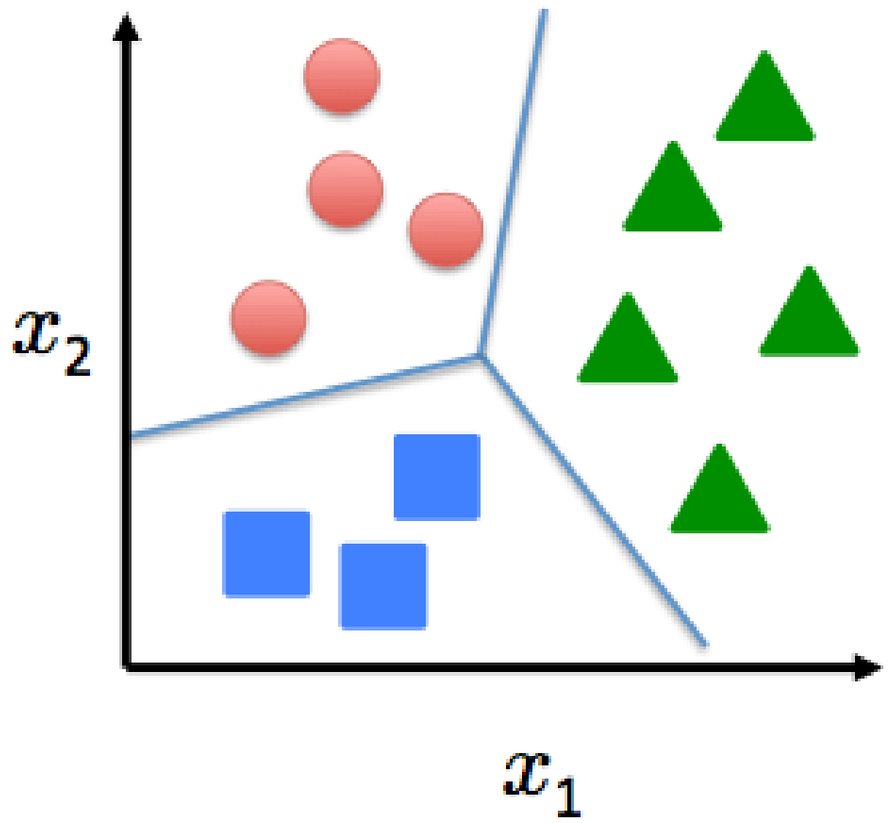
- Generative and Discriminative Classification
- The Logistic Regression Model
- Understanding the Objective Function
- Gradient Descent for Parameter Learning
- Multiclass Logistic Regression 

Multiclass Logistic Regression

Binary classification:



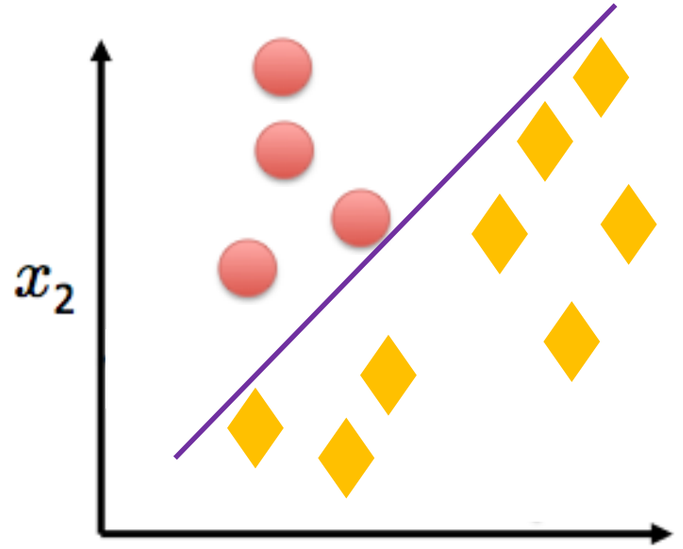
Multi-class classification:



Disease diagnosis: healthy / cold / flu / pneumonia

Object classification: desk / chair / monitor / bookcase

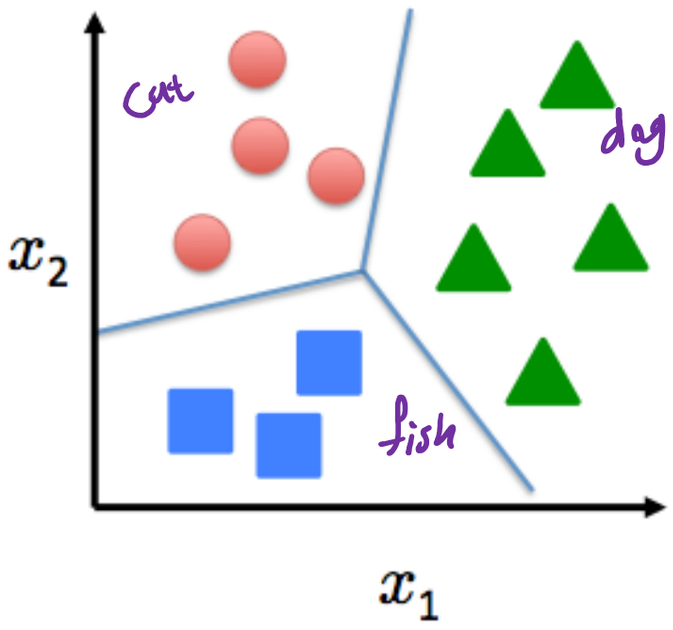
One-vs-all (one-vs-rest)

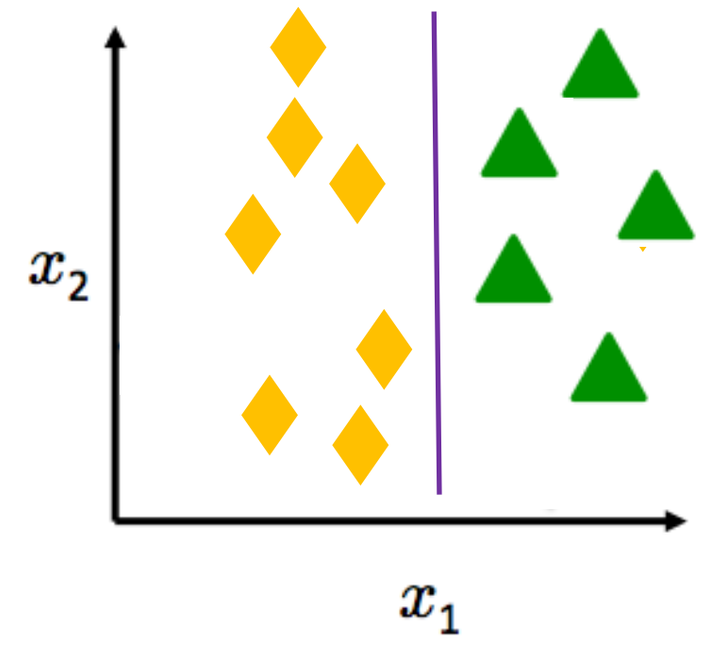


$h_\theta^1(x)$

0.7

Multi-class classification:

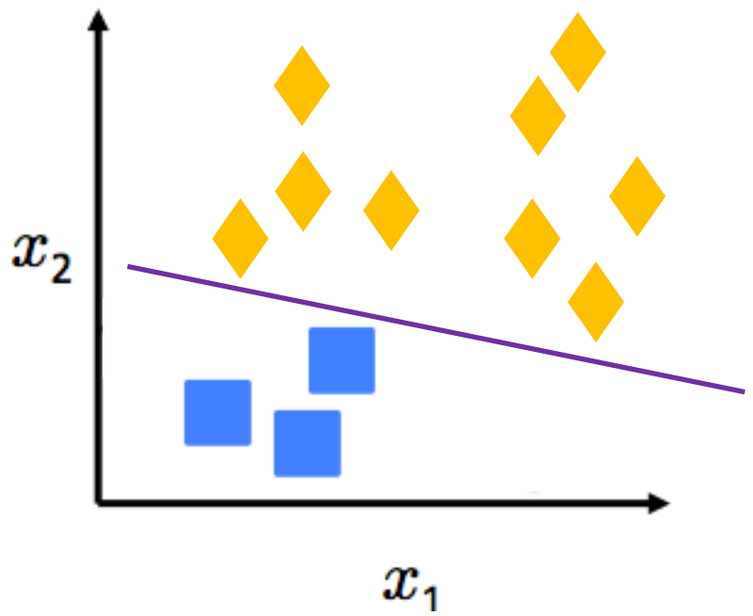




$h_\theta^2(x)$

0.8

$h_\theta^{(m)}(x) = p(y = 1|x, \theta)\ (m = 1,2,3)$



$h_\theta^3(x)$

0.5

One-vs-all (one-vs-rest)

Train a logistic regression $h_\theta^{(m)}(x)$ for each class m

To predict the label of a new input x, pick class m that maximizes:

$$\max_i h_\theta^{(m)}(x)$$

Using Softmax

$$L(\theta) = -\sum_{i=1}^{N} y_a^{\{i\}} * \log\left(y_p^{\{i\}}\right)$$

$y_a = [cat, dog, fish] = [1,0,0]$
$\Rightarrow there\ are\ M\ classes\ (M = 3\ in\ this\ example)$

$$y_{p\ for\ class\ \boldsymbol{m}} = softmax(x\theta) = \frac{\exp(x\theta)_{\boldsymbol{m}}}{\sum_{j=0}^{M} \exp(x\theta)_j}$$

$$y_p = [0.6,0.3,0.1]$$

$$SGD \Rightarrow \theta^{t+1} \leftarrow \theta^t - \alpha\, \nabla L(\theta)$$

$$\theta^{t+1} \leftarrow \theta^t - \alpha\, x^T (y_p - y_a)$$

Take-Home Messages

- Generative and Discriminative Classification
- The Logistic Regression Model
- Understanding the Objective Function
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- Multiclass Logistic Regression