

Naïve Bayes and Logistic Regression

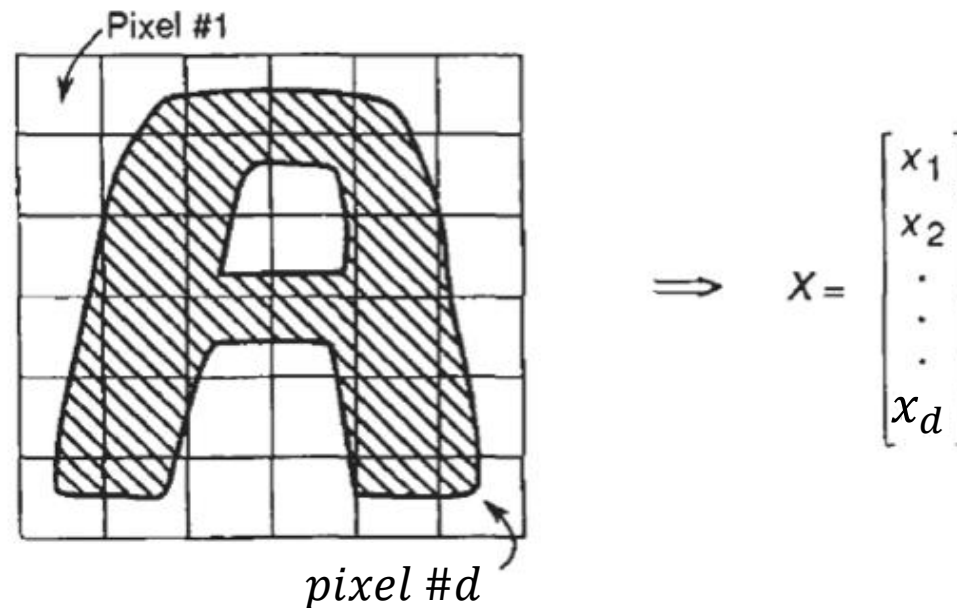
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Outline

- Generative and Discriminative Classification 
- The Logistic Regression Model
- Understanding the Objective Function
- Gradient Descent for Parameter Learning
- Multiclass Logistic Regression

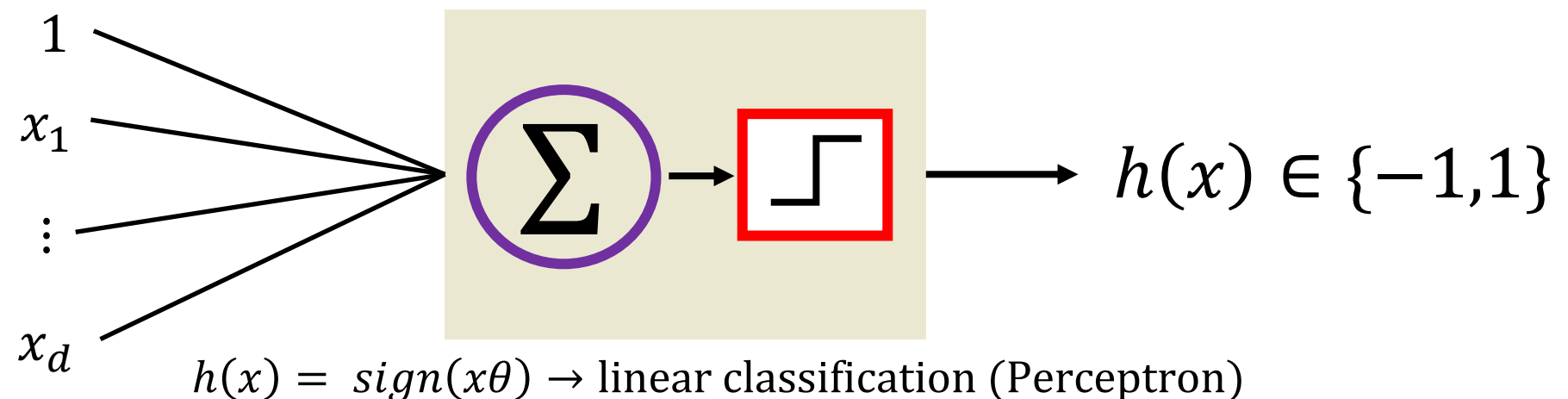
Classification

- Represent the data



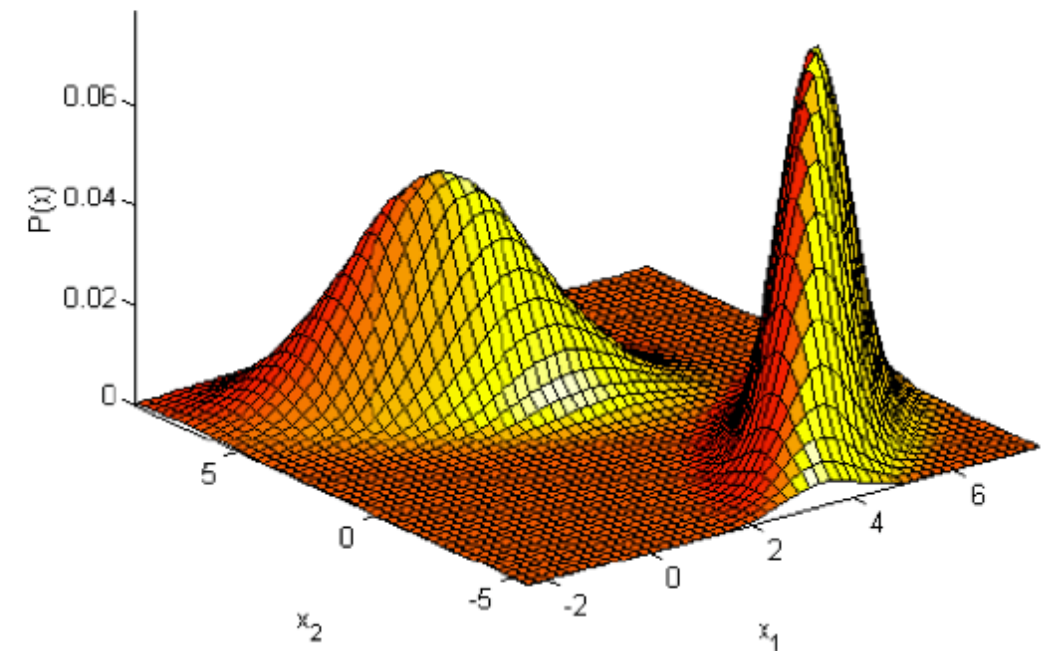
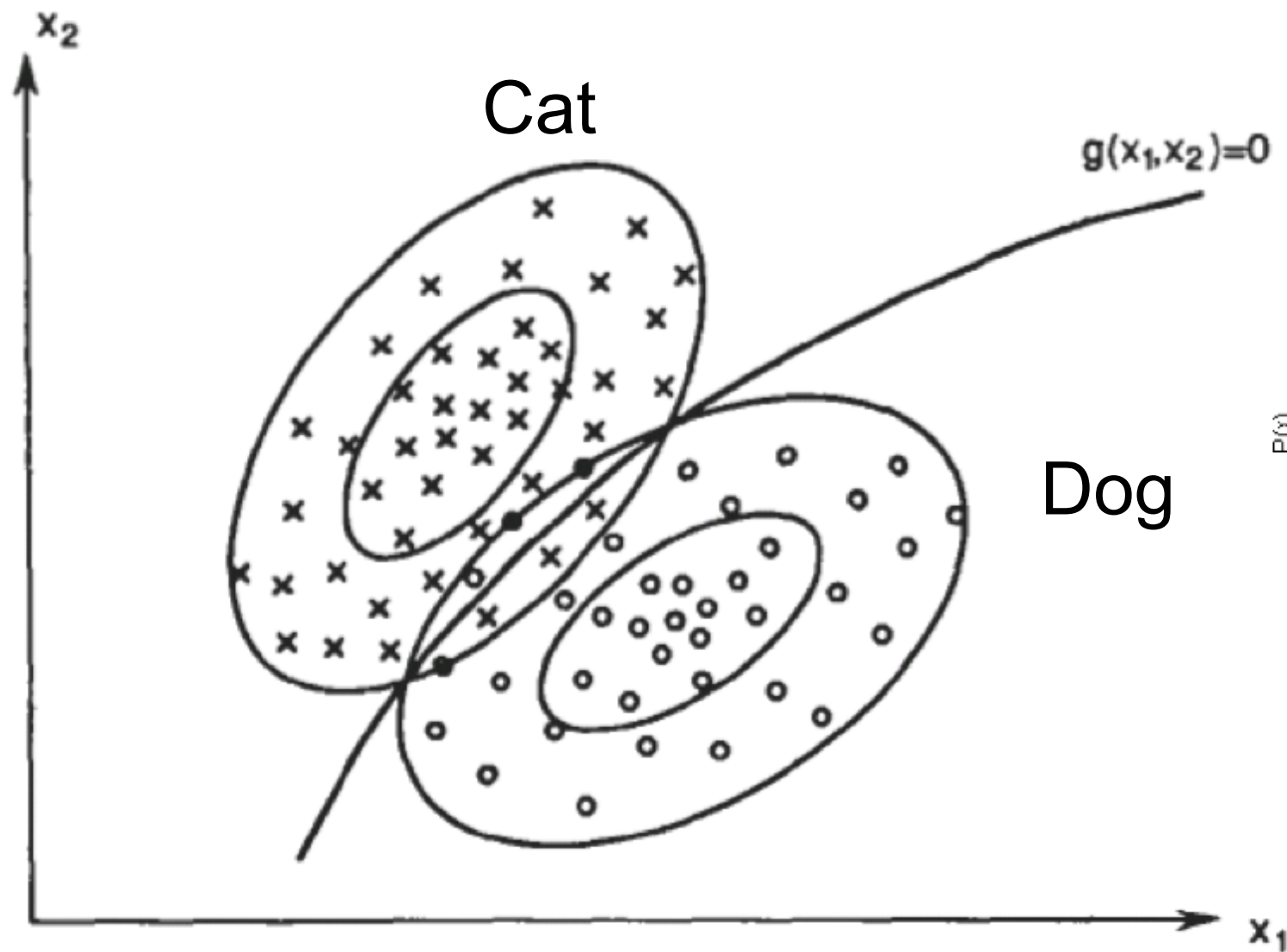
- A label is provided for each data point, eg., $y \in \{-1, +1\}$

- Classifier



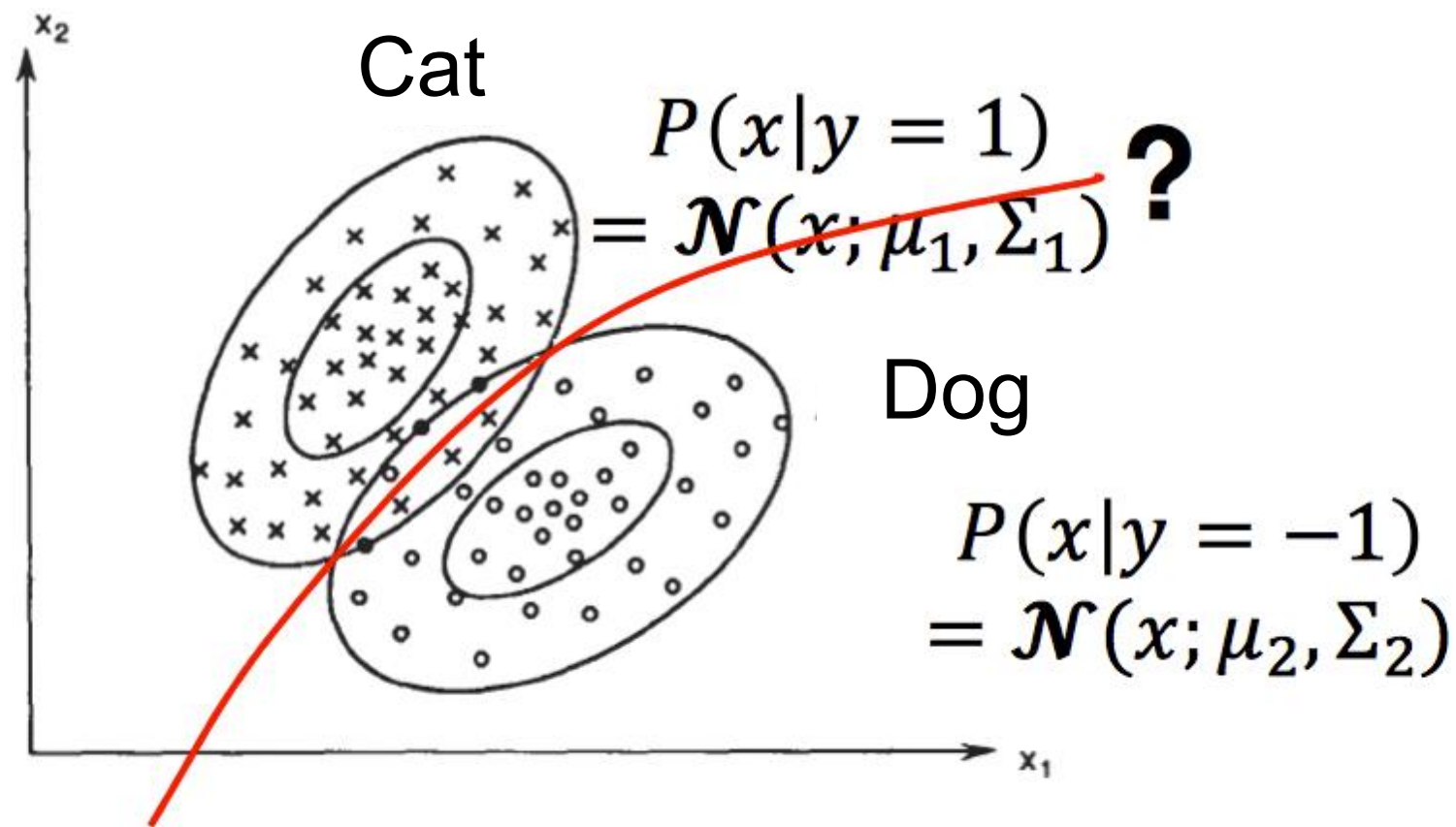
Decision Making: Dividing the Feature Space

- Distributions of sample from normal (positive class) and abnormal (negative class) tissues



How to Determine the Decision Boundary?

- Given class conditional distribution: $P(x|y = 1)$, $P(x|y = -1)$, and class prior: $P(y = 1)$, $P(y = -1)$



Bayes Decision Rule

likelihood

Prior

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x, y)}{\sum_z P(x, y)}$$

posterior

normalization constant

Prior: $P(y)$

Likelihood (class conditional distribution : $p(x|y) = \mathcal{N}(x|\mu_y, \Sigma_y)$

Posterior: $P(y|x) = \frac{P(y)\mathcal{N}(x|\mu_y, \Sigma_y)}{\sum_y P(y)\mathcal{N}(x|\mu_y, \Sigma_y)}$

Bayes Decision Rule

- Learning: prior: $p(y)$, class conditional distribution : $p(x|y)$

- The poster probability of a test point

$$q_i(x) := P(y = i|x) = \frac{P(x|y)P(y)}{P(x)}$$

- Bayes decision rule:

- If $q_i(x) > q_j(x)$, then $y = i$, otherwise $y = j$

- Alternatively:

- If ratio $l(x) = \frac{P(x|y=i)}{P(x|y=j)} > \frac{P(y=j)}{P(y=i)}$, then $y = i$, otherwise $y = j$

- Or look at the log-likelihood ratio $h(x) = -\ln \frac{q_i(x)}{q_j(x)}$

What do People do in Practice?

- Generative models
 - Model prior and likelihood explicitly
 - “Generative” means able to generate synthetic data points
 - Examples: Naive Bayes, Hidden Markov Models
- Discriminative models
 - Directly estimate the posterior probabilities
 - No need to model underlying prior and likelihood distributions
 - Examples: Logistic Regression, SVM, Neural Networks

Generative Model: Naive Bayes

- Use Bayes decision rule for classification

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

- But assume $p(x|y = 1)$ is fully factorized: Dimensions are independent.

$$p(x|y = 1) = \prod_{i=1}^d p(x_i|y = 1)$$

- Or the variables corresponding to each dimension of the data are independent given the label

“Naïve” conditional independence assumption

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)} = \frac{P(x, y)}{P(x)}$$

Joint probability model:

$$P(x, y_{label=1}) = P(x_1, \dots, x_d, y_{label=1}) = P(x_1 | x_2, \dots, x_d, y_{label=1}) P(x_2, \dots, x_d, y_{label=1})$$


$$= P(x_1 | x_2, \dots, x_d, y_{label=1}) P(x_2 | x_3, \dots, x_d, y_{label=1}) P(x_3, \dots, x_d, y_{label=1})$$
$$= \dots$$

$$= P(x_1 | x_2, \dots, x_d, y_{label=1}) P(x_2 | x_3, \dots, x_d, y_{label=1}) \dots P(x_{d-1} | x_d, y_{label=1}) P(x_d | y_{label=1}) P(y_{label=1})$$

Naïve Bayes assumption: let's rewrite it as:

$$P(x, y_{label=1}) = P(x_1 | y_{label=1}) P(x_2 | y_{label=1}) \dots P(x_d | y_{label=1}) P(y_{label=1}) =$$

$$P(y_{label=1}) \prod_{i=1}^d P(x_i | y_{label=1})$$

Gaussian naïve Bayes
A typical assumption

Example

Discriminative Models

- Directly estimate decision boundary $h(x) = -\ln \frac{q_i(x)}{q_j(x)}$ or posterior distribution $p(y|x)$
 - Logistic regression, Neural networks
 - Do not estimate $p(x|y)$ and $p(y)$
- Why discriminative classifier?
 - Avoid difficult density estimation problem  Generative model
 - Empirically achieve better classification results

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Gaussian Naïve Bayes

$$P(y = 1|x) = \frac{P(x|y = 1)P(y = 1)}{P(x)} = \frac{P(y = 1) \prod_{i=1}^d P(x_i|y = 1)}{P(x)}$$

$$\begin{aligned} \prod_{i=1}^d p(x_i|y = 1, \mu_{1i}, \sigma_{1i}) \\ = \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_{1i}} \exp\left(-\frac{1}{2\sigma_{1i}^2} (x_{1i} - \mu_{1i})^2\right) \end{aligned}$$

Prior: $p(y = 1) = \pi_1$

Posterior: $p(y = 1 | x, \mu, \sigma, \pi)$

$$= \frac{\pi_1 \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_{1i}} \exp\left(-\frac{1}{2\sigma_{1i}^2} (x_i - \mu_{1i})^2\right)}{\sum_{\substack{k=1 \\ \text{labels}}}^2 \pi_k \prod_{i=1}^d \frac{1}{\sqrt{2\pi}\sigma_{ki}} \exp\left(-\frac{1}{2\sigma_{ki}^2} (x_i - \mu_{ki})^2\right)}$$

get $\exp(\ln(u))$ of numerator and denominator

$$= \frac{\exp\left(-\sum_{i=1}^d \left(\frac{1}{2\sigma_{1i}^2} (x_i - \mu_{1i})^2 + \log \sigma_{1i} + C\right) + \log \pi_1\right)}{\sum_{k=1}^2 \exp\left(-\sum_{i=1}^d \left(\frac{1}{2\sigma_{ki}^2} (x_i - \mu_{ki})^2 + \log \sigma_{ki} + C\right) + \log \pi_k\right)}$$

$$= \frac{\exp \left(-\sum_{i=1}^d \left(\frac{1}{2\sigma_i^2} (x_i - \mu_{1i})^2 + \log \sigma_i + C \right) + \log \pi_1 \right)}{\sum_{k=1}^2 \exp \left(-\sum_{i=1}^d \left(\frac{1}{2\sigma_i^2} (x_i - \mu_{ki})^2 + \log \sigma_i + C \right) + \log \pi_k \right)}$$

$$= \frac{1}{1 + \exp \left(\underbrace{-\sum_{i=1}^d \left(x_i \frac{1}{\sigma_i} (\mu_{1i} - \mu_{2i}) \right)}_{\sum_i \theta_i x_i} + \underbrace{\frac{1}{\sigma_i^2} (\mu_{1i}^2 - \mu_{2i}^2)}_{\theta_0} \right) + \log \frac{\pi_2}{\pi_1} \right)}$$

$$P(y = 1|x) = \frac{1}{1 + \exp \left(- \sum_{i=1}^d \left(x_i \frac{1}{\sigma_i} (\mu_{1i} - \mu_{2i}) + \frac{1}{\sigma_i^2} (\mu_{1i}^2 - \mu_{2i}^2) \right) + \log \frac{\pi_2}{\pi_1} \right)}$$

Number of parameters:

$2d + 1 \rightarrow$ *d mean, d variance, and 1 for prior*

$$P(y = 1|x) = \frac{1}{1 + \exp[-(\sum_{i=1}^d (\theta_i x_i) + \theta_0)]} = \frac{1}{1 + \exp(-s)}$$

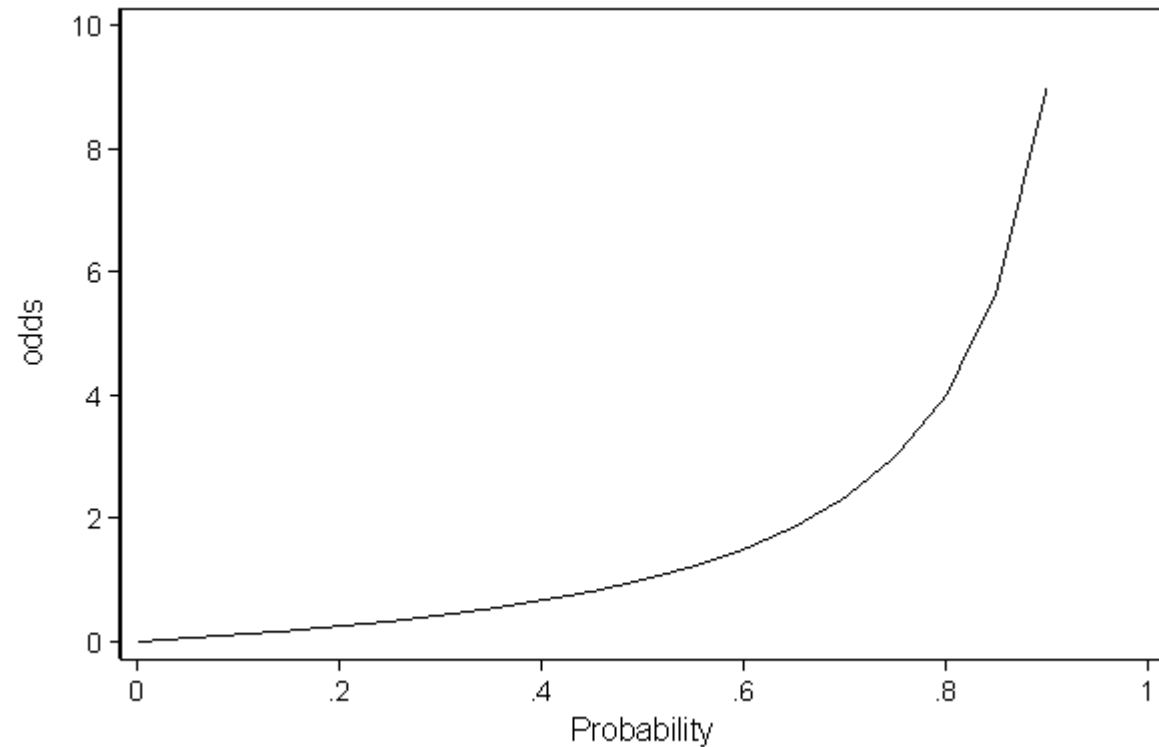
Number of parameters = $d + 1 \rightarrow \theta_0, \theta_1, \theta_2, \dots, \theta_d$

Why not directly learning $P(y = 1|x)$ or θ parameters?

Gaussian Naïve Bayes is a subset of logistic regression

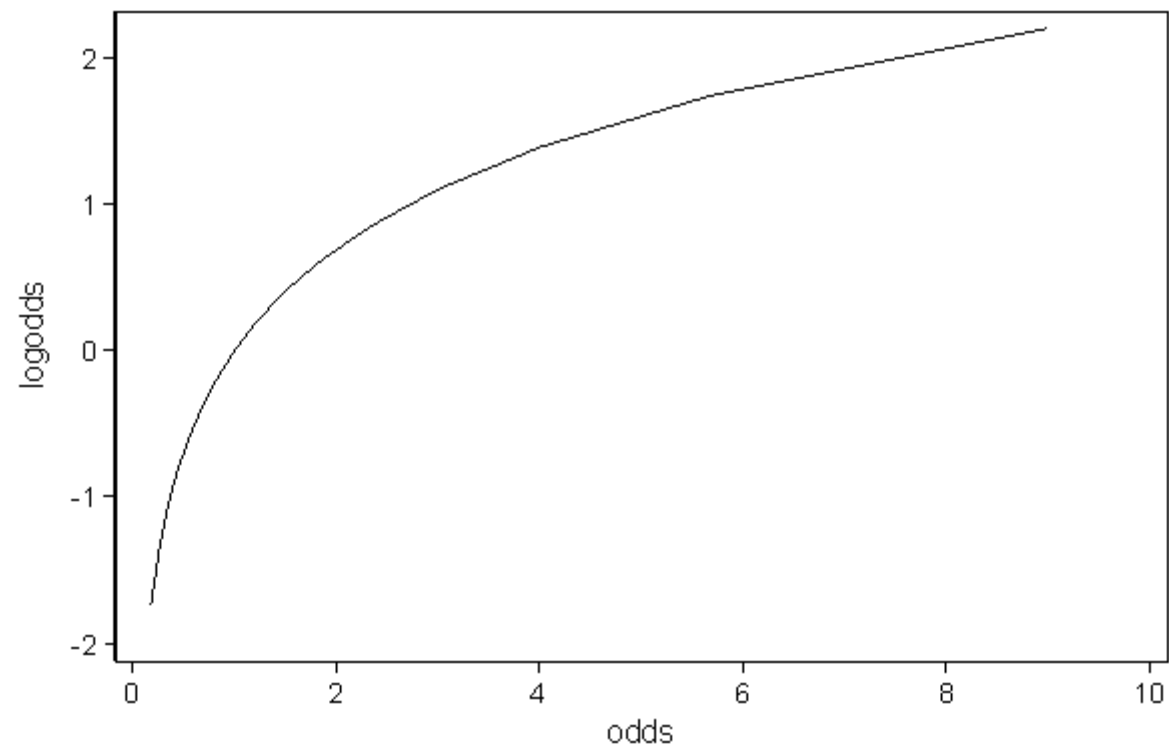
Why $\frac{1}{1+\exp(-x\theta)}$ is a probability?

$\frac{P(y = 1|x)}{1-P(y = 1|x)}$ is called Odds



log(odds) vs odds

What could be $x\theta$ domain?



What is logit function?

$$\text{logit}(p) = \log(\text{odds}) = \log\left(\frac{p}{1-p}\right)$$

$$\log\left(\frac{p}{1-p}\right) = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d = \sum_{i=0}^d x_i \theta_i = x\theta$$

$$\exp\left(\log\left(\frac{p}{1-p}\right)\right) = \exp(x\theta)$$

$$p = \frac{e^{x\theta}}{1 + e^{x\theta}} = \frac{1}{1 + e^{-x\theta}}$$

Logistic function for posterior probability

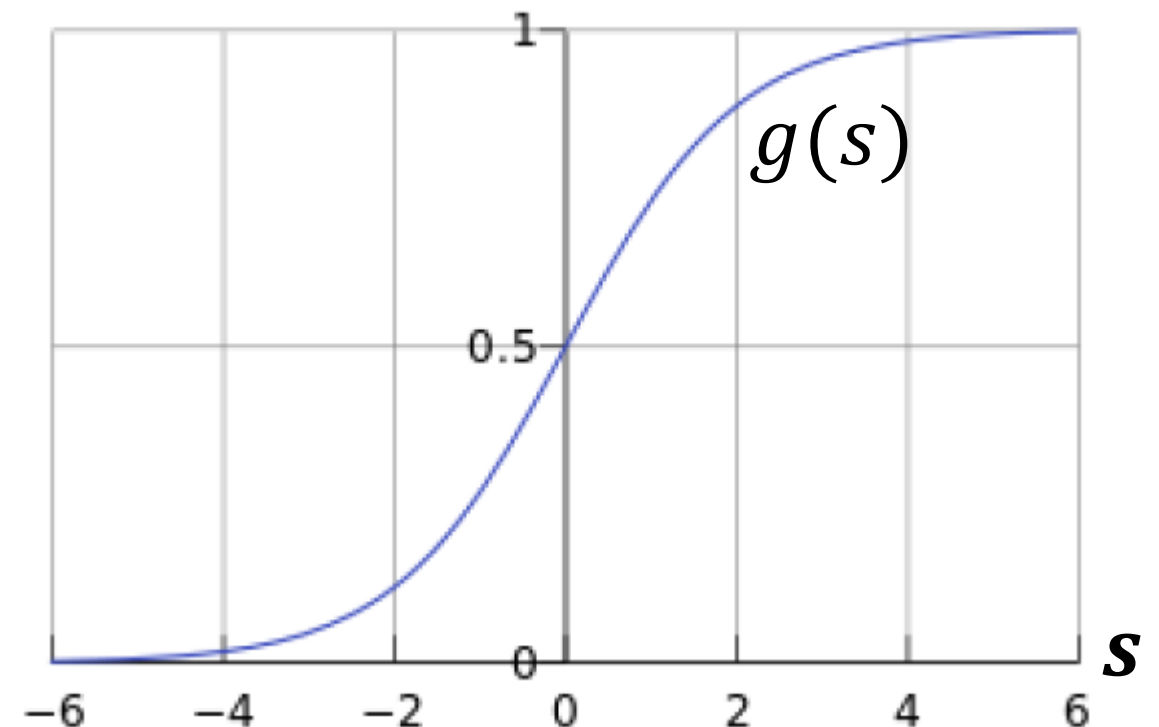
Many equations can give us this shape

Let's use the following function:

$$s = x\theta$$

$$g(s) = P(y = 1|x) = \frac{e^s}{1 + e^s} = \frac{1}{1 + e^{-s}}$$

This formula is called sigmoid function



It is easier to use this function for optimization

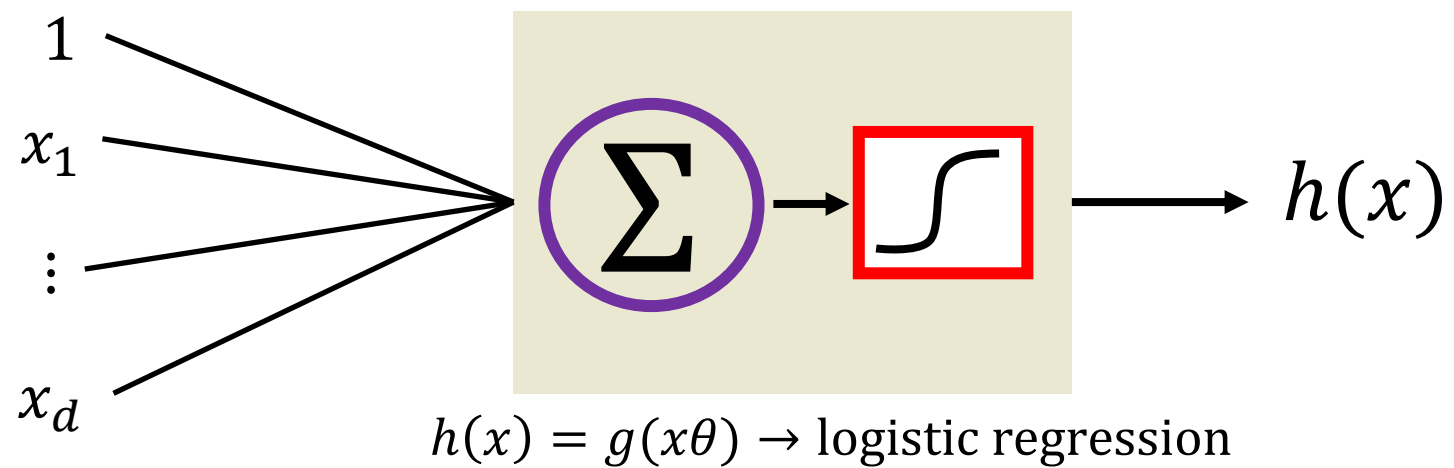
Is 0.5 threshold cut-off a good choice?

[Learn about ROC and AUC \(False positive rate and True positive rate\) \(Interactive\)](#)

$$g(s) = \frac{e^s}{1 + e^s} = \frac{1}{1 + e^{-s}}$$

Sigmoid Function

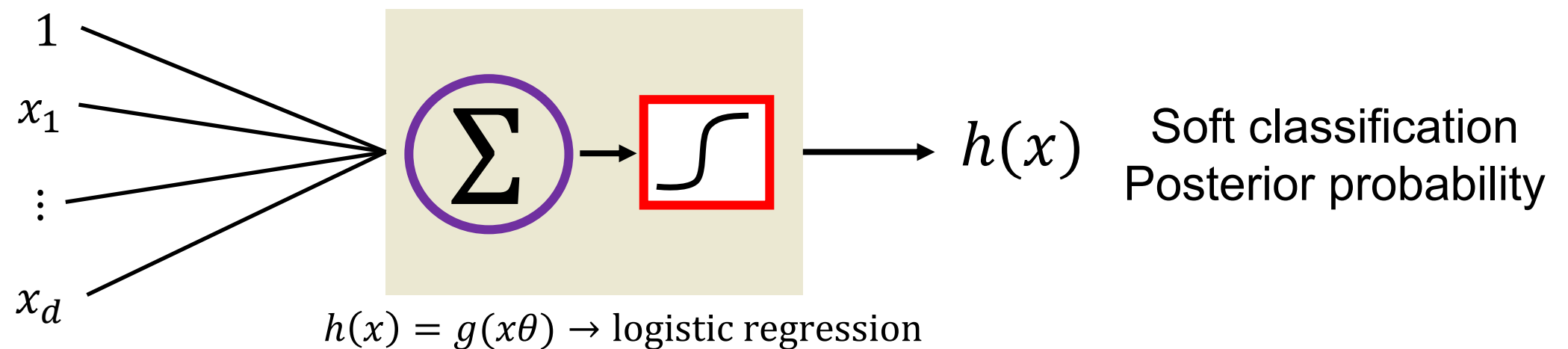
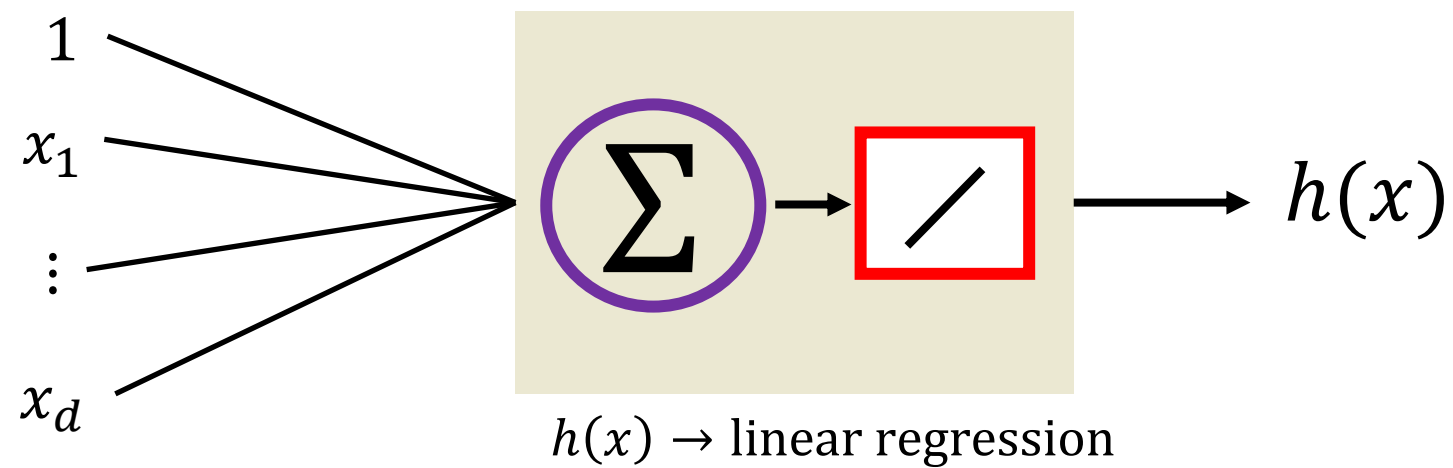
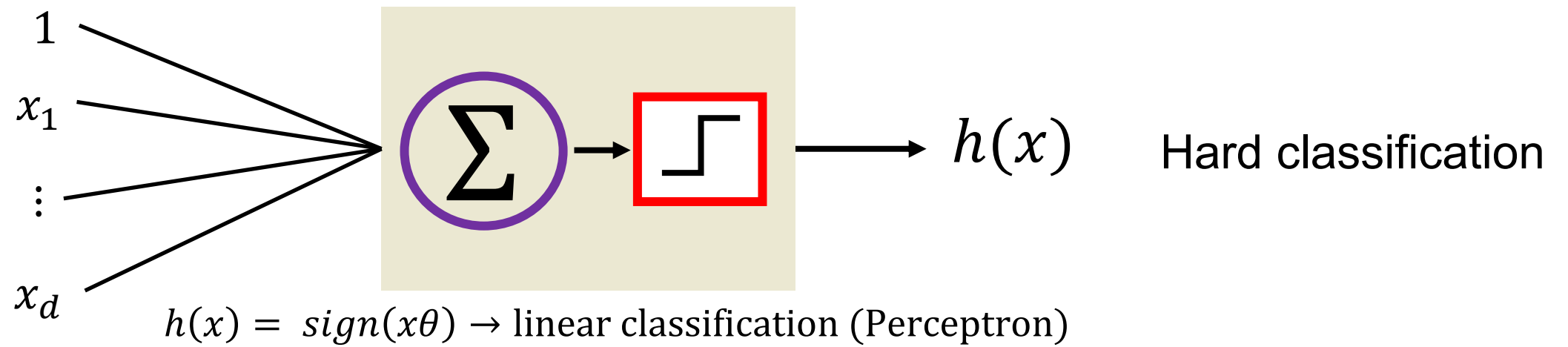
$$s = \sum_{i=0}^d x_i \theta_i = \theta_0 + \theta_1 x_1 + \cdots + \theta_d x_d$$



Soft classification
Posterior probability

$$s = \sum_{i=0}^d x_i \theta_i = \theta_0 + \theta_1 x_1 + \dots + \theta_d x_d$$

Three linear models



$g(s)$ is interpreted as probability

Example: Prediction of heart attacks

Input x : cholesterol level, age, weight, finger size, etc.

$g(s)$: probability of heart attack within a certain time

We can't have a hard prediction here

$s = x\theta$ Let's call this risk score

$$h_{\theta}(x) = p(y|x) = \begin{cases} g(s), & y = 1 \\ 1 - g(s), & y = 0 \end{cases}$$

Using posterior probability directly

Logistic regression model

$$p(y|x) = \begin{cases} \frac{1}{1 + \exp(-x\theta)} & y = 1 \\ 1 - \frac{1}{1 + \exp(-x\theta)} = \frac{\exp(-x\theta)}{1 + \exp(-x\theta)} & y = 0 \end{cases}$$

We need to find θ parameters, let's set up log-likelihood for n datapoints

$$\begin{aligned} l(\theta) &:= \log \prod_{i=1}^n p(y^{\{i\}} | x^{\{i\}}, \theta) \\ &= \sum_i \theta^T (x^{\{i\}})^T (y^{\{i\}} - 1) - \log(1 + \exp(-x^{\{i\}} \theta)) \end{aligned}$$

This form is concave, negative of this form is convex

The gradient of $l(\theta)$

$$l(\theta) = \log \prod_{i=1}^n p(y^{\{i\}} | x^{\{i\}}, \theta)$$
$$= \sum_i \theta^T (x^{\{i\}})^T (y^{\{i\}} - 1) - \log(1 + \exp(-x^{\{i\}} \theta))$$

- Gradient

$$\frac{\partial l(\theta)}{\partial \theta} = \sum_i (x^{\{i\}})^T (y^{\{i\}} - 1) + (x^{\{i\}})^T \frac{\exp(-x^{\{i\}} \theta)}{1 + \exp(-x^{\{i\}} \theta)}$$

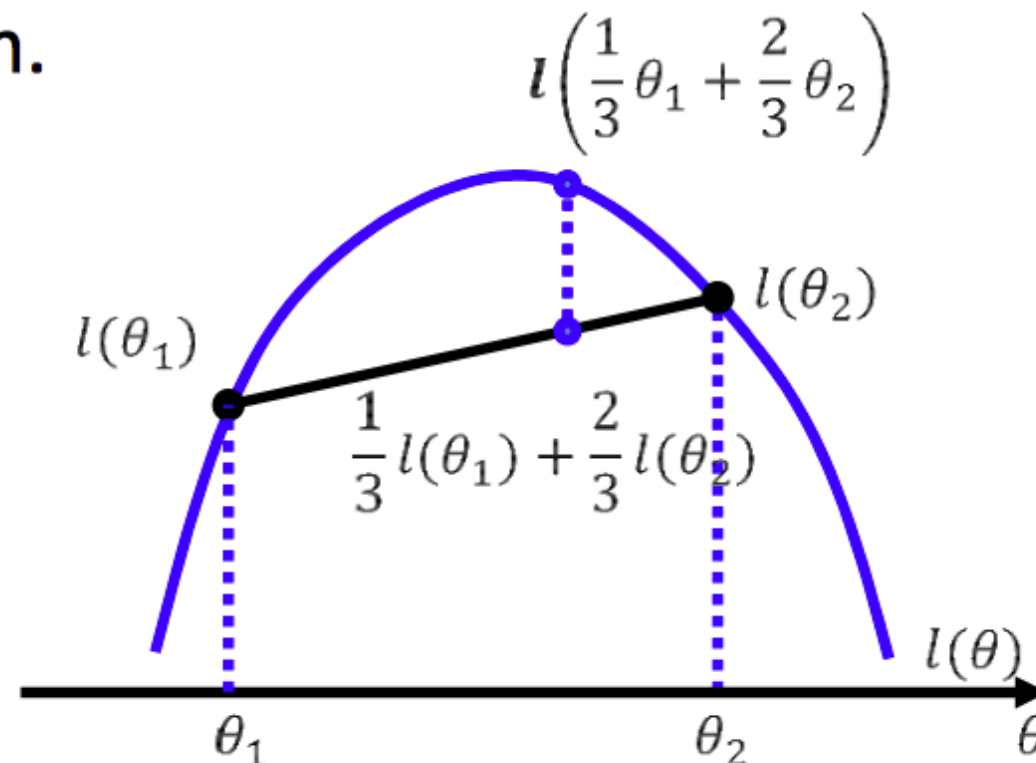
- Setting it to 0 does not lead to closed form solution

The Objective Function

- Find θ , such that the conditional likelihood of the labels is maximized

$$\max_{\theta} l(\theta) := \log \prod_{i=1}^{\bar{n}} p(y^{\{i\}} | x^{\{i\}}, \theta)$$

- Good news: $l(\theta)$ is concave function of θ , and there is a single global optimum.



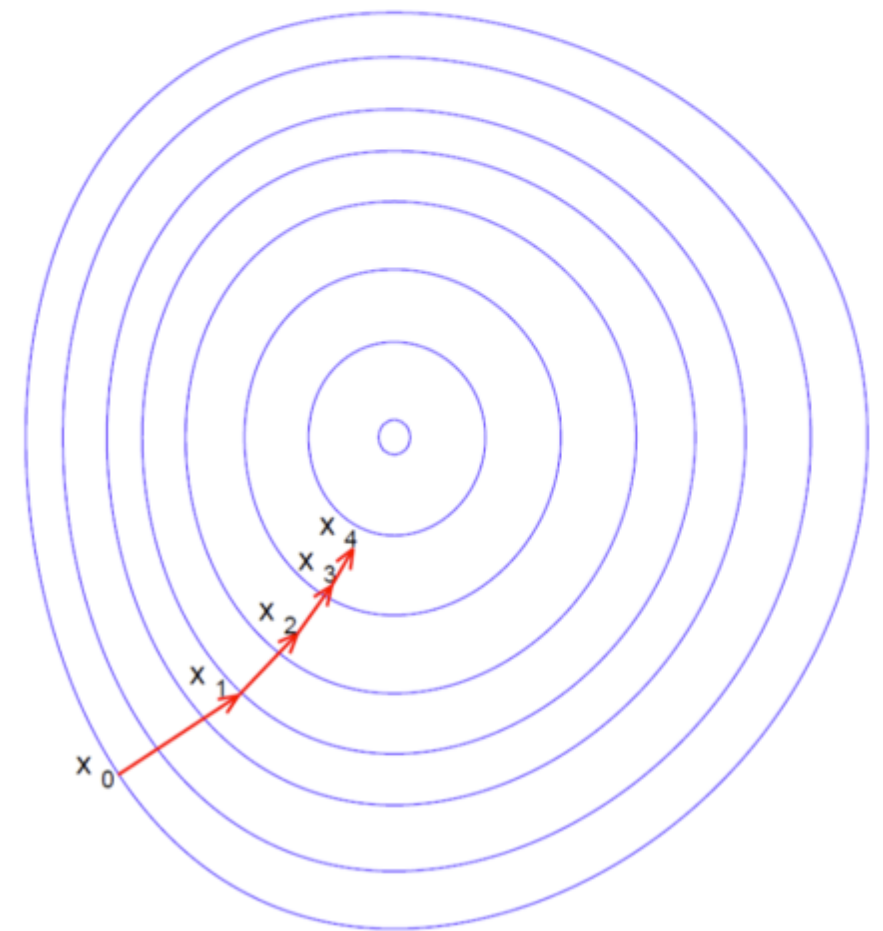
- Bad new: no closed form solution (resort to numerical method)

Gradient Descent

- One way to solve an *unconstrained* optimization problem is gradient descent
- Given an initial guess, we *iteratively* refine the guess by taking the direction of the negative gradient
- Think about going down a hill by taking the steepest direction at each step
- Update rule

$$x_{k+1} = x_k - \gamma_k \nabla f(x_k)$$

γ_k is called the step size or learning rate



Gradient Ascent(concave)/Descent(convex) algorithm

- Initialize parameter θ^0

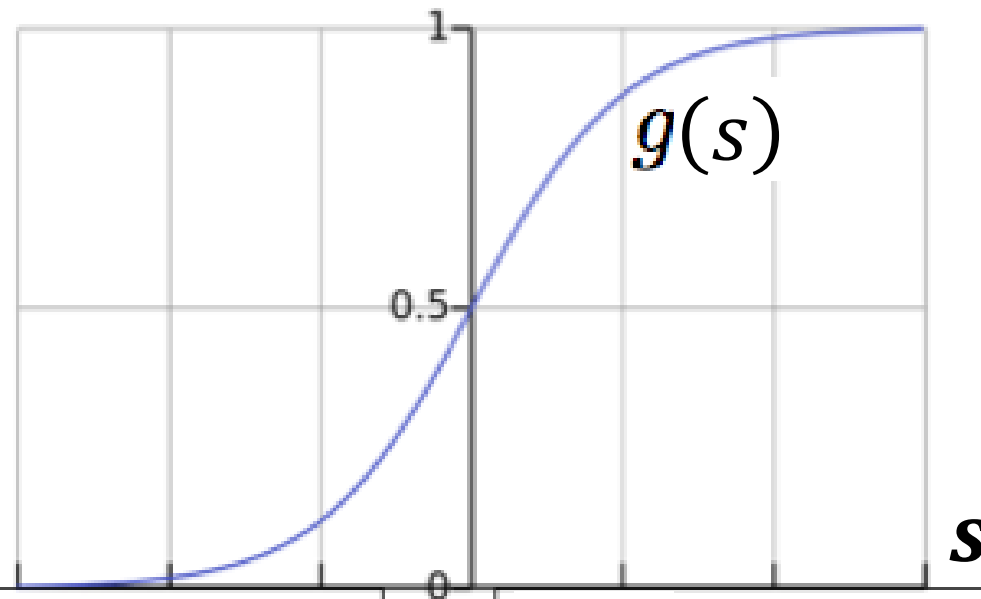
- Do

$$\theta^{t+1} \leftarrow \theta^t + \eta \sum_i (x^{\{i\}})^T (y^{\{i\}} - 1) + (x^{\{i\}})^T \frac{\exp(-x^{\{i\}}\theta)}{1 + \exp(-x^{\{i\}}\theta)}$$

- While the $||\theta^{t+1} - \theta^t|| > \epsilon$

Logistic Regression

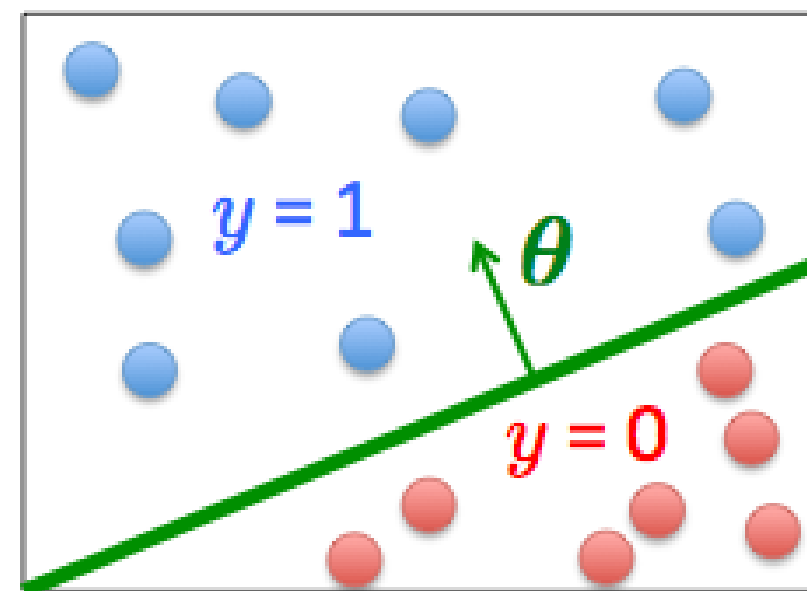
$$g(s) = \frac{e^s}{1 + e^s} = \frac{1}{1 + e^{-s}}$$
$$s = x\theta$$



$x\theta$ should be large negative
values for negative instances

$x\theta$ should be large positive
values for positive instances

- Assume a threshold and...
 - Predict $y = 1$ if $g(s) \geq 0.5$
 - Predict $y = 0$ if $g(s) < 0.5$

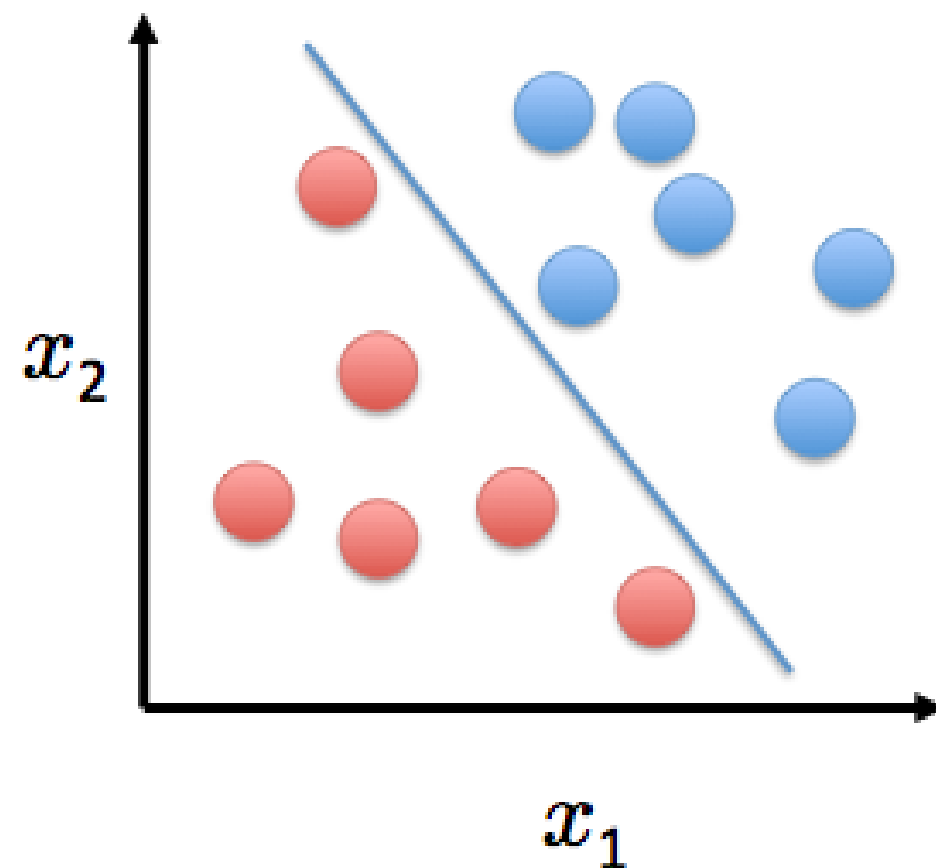


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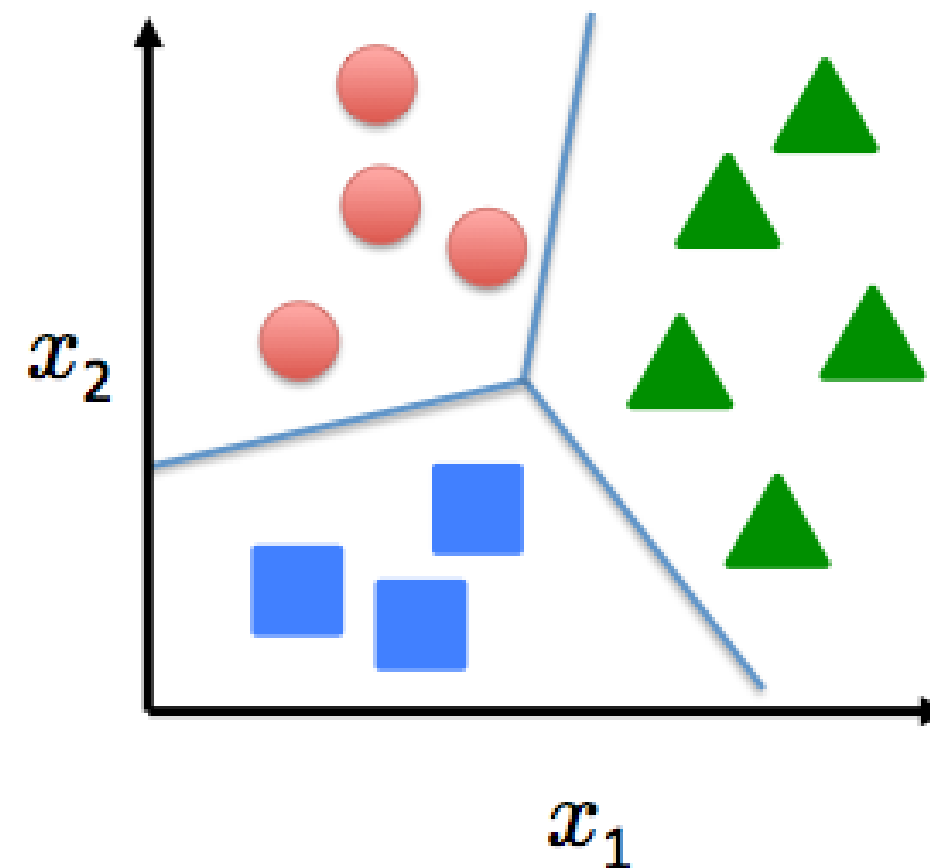
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Multiclass Logistic Regression

Binary classification:



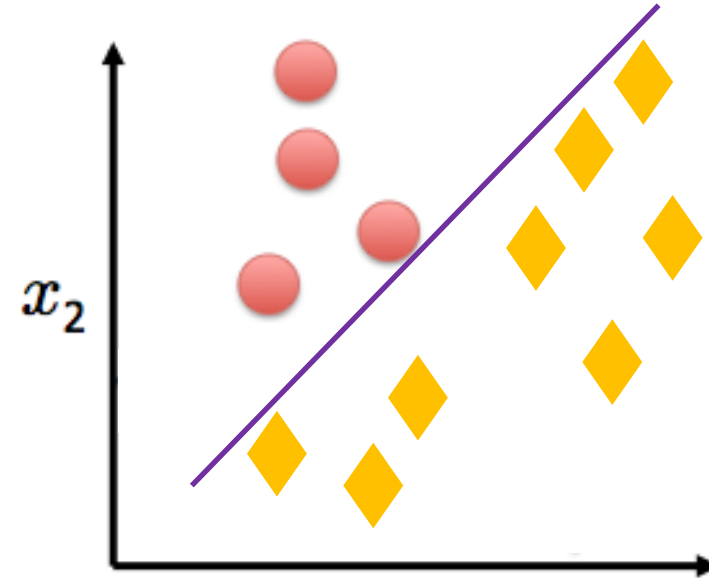
Multi-class classification:



Disease diagnosis: healthy / cold / flu / pneumonia

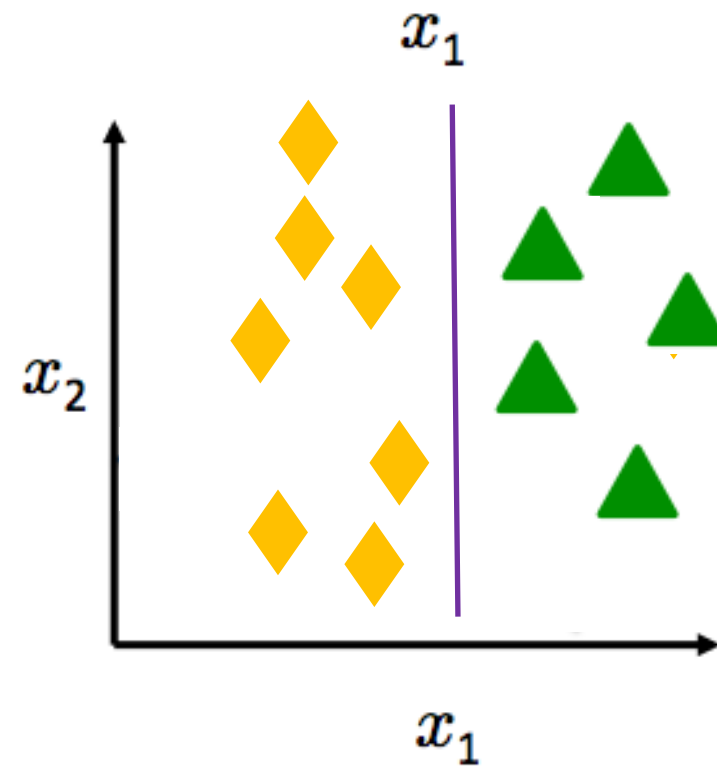
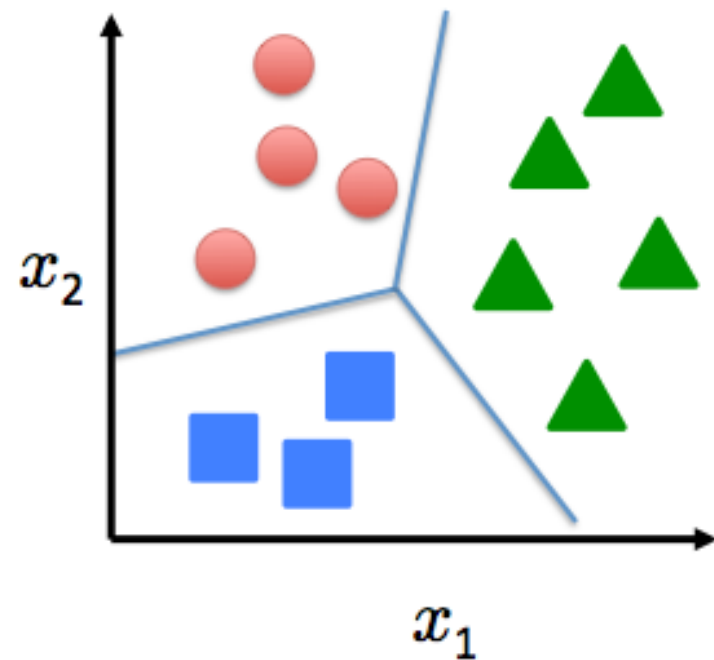
Object classification: desk / chair / monitor / bookcase

One-vs-all (one-vs-rest)

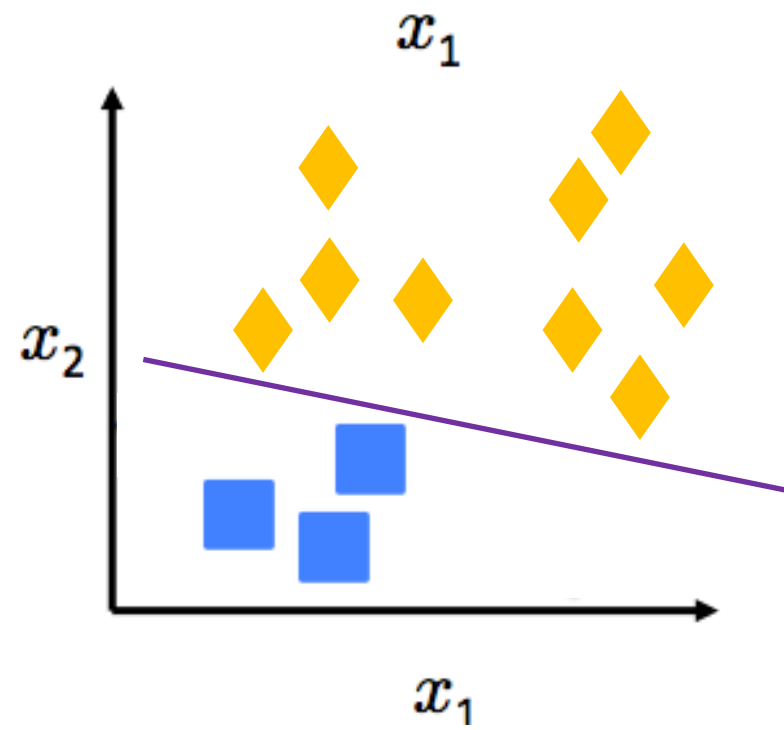


$$h_{\theta}^1(x)$$

Multi-class classification:



$$h_{\theta}^2(x)$$



$$h_{\theta}^3(x)$$

$$h_{\theta}^{(m)}(x) = p(y = 1|x, \theta) \quad (m = 1, 2, 3)$$

One-vs-all (one-vs-rest)

Train a logistic regression $h_{\theta}^{(m)}(x)$ for each class m

To predict the label of a new input x , pick class m that maximizes:

$$\max_i h_{\theta}^{(m)}(x)$$

Using Softmax

$$L(\theta) = - \sum_{i=1}^N y_a^{\{i\}} * \log(y_p^{\{i\}})$$

$$y_a = [cat, dog, fish] = [1, 0, 0]$$

$\Rightarrow$ there are M classes ($M = 3$ in this example)

$$y_p \text{ for class } \mathbf{m} = \text{softmax}(x\theta) = \frac{\exp(x\theta)_{\mathbf{m}}}{\sum_{j=0}^M \exp(x\theta)_j}$$

$$y_p = [0.6, 0.3, 0.1]$$

$$SGD \Rightarrow \theta^{t+1} \leftarrow \theta^t - \alpha \nabla L(\theta)$$

$$\theta^{t+1} \leftarrow \theta^t - \alpha x^T (y_p - y_a)$$

Take-Home Messages

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